IMO Aftermath

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1. (IMO 2010, Problem 1) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality
$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$
holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

2. Prove that any arithmetic progression $a, a+d, a+2d, a+3d, \ldots$ (a and d are real) contains only finitely many elements x such that $0 \leq x \leq 1$.

3. Suppose that eggs are being sold in cartons of $1, 2, \ldots, s$ with respective prices $a_1, a_2, \ldots, a_s$. Prove that there is an integer ℓ ($1 \leq \ell \leq s$) with the following property: for all large enough n, the cheapest price of buying exactly n eggs can be achieved with a combination that includes at least one ℓ-egg carton.

4. (IMO 2010, Problem 6) Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have
$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\}$$
for all $n > s$. Prove that there exist positive integers ℓ and N, with $\ell \leq s$ and such that $a_n = a_\ell + a_{n-\ell}$ for all $n \geq N$.