Formulas for Complex Numbers in Geometry

Evan O'Dorney

November 18, 2010

1 Transformations

- The map $z \mapsto z + 1$ represents a **translation** by one unit to the right.
- The map $z \mapsto z + a$ (a is a complex number) represents the **translation** that sends the point 0 to a.
- The map $z \mapsto 2z$ represents a **dilation** which stretches the entire plane by a factor of 2 away from 0.
- The map $z \mapsto iz$ represents a **rotation** by 90° about 0.
- The map $z \mapsto -z$ represents a **rotation** by 180° about 0.
- The map $z \mapsto \bar{z}$ represents a **reflection** about the x-axis.
- The map $z \mapsto -\bar{z}$ represents a **reflection** about the y-axis.
- The map $z \mapsto 1/\bar{z}$ represents an **inversion** about the unit circle (for those who have studied inversion).

2 Equations

- The equation $z = \bar{z}$ holds if and only if z is real.
- The equation $z = -\bar{z}$ holds if and only if z is pure imaginary.
- The equation $z\bar{z} = 1$ holds if and only if z is on the unit circle.

3 Distances

- For any complex number z, $|z|$ is the distance from z to the origin.
- $|z|^2 = z\bar{z}$.
- The distance from z to w is $|z - w|$.

4 Lines

- If $a \neq 0$, then a and b lie on the same line through the origin iff b/a is real, that is, $b/a = \bar{b}/\bar{a}$.
- If a, b, c are distinct, then they are collinear iff

$$\frac{c-a}{b-a} \text{ is real, that is, } \quad \frac{c-a}{b-a} = \frac{\bar{c}-\bar{a}}{\bar{b}-\bar{a}}.$$

- If a and b are on the unit circle, the equation of "chord" AB (really the secant line) is $z = a + b - ab\bar{z}$. If $a = b$, this is the equation of the tangent line at a.
- If a, b, c, and d are on the unit circle, the intersection E of lines AB and CD is given by

$$\bar{e} = \frac{a+b-c-d}{ab-cd} \quad \text{or} \quad e = \frac{abc+abd-acd-bcd}{ab-cd}.$$

  Either or both of the lines may be tangent lines.

5 Angles

- If AOB and COD are two angles with both counterclockwise (or both clockwise) orientation, then they are equal iff

$$\frac{d}{c} \div \frac{b}{a}, \quad \text{that is,} \quad \frac{ad}{bc}$$

  is real.

- If BAC and EDF are two angles with both counterclockwise (or both clockwise) orientation, then they are equal iff

$$\frac{(b-a)(f-d)}{(c-a)(e-d)}$$

  is real.

- Four points a, b, c, d are collinear iff the "cross-ratio"

$$\frac{(d-c)(b-a)}{(d-b)(c-a)}$$

  is real.