Probability notes and problems

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1 Quick Reference

- Set notation for events

$$\cap \qquad \text{and, intersect} \tag{1}$$
$$\cup \qquad \text{or, union} \tag{2}$$
$$\sim A \qquad \text{not A, complement of } A \tag{3}$$

- Sum rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{4}$$

- Definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{5}$$

or, equivalently,

$$P(A \cap B) = P(B)P(A|B) \tag{6}$$

- Definition of odds

$$\text{odds}(A) = \frac{P(A)}{1 - P(A)} \tag{7}$$

- Bayes rule

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)} \tag{8}$$

- Bayes rule in terms of odds

$$\text{odds}(A|B) = \text{odds}(A)\frac{P(B|A)}{P(B|\sim A)} \tag{9}$$

2 Bayes Rule

Bayes rule follows from the definition of conditional probability.

$$P(A|B) = \frac{P(A \cup B)}{P(B)} \qquad \text{using 5} \tag{10}$$
$$= \frac{P(A)P(B|A)}{P(B)} \qquad \text{using 6} \tag{11}$$

Here's an archetypical Bayes rule problem. I have two coins, a fair coin and a biased coin. The fair coin lands on heads with probability .5. The biased coin lands on heads with probability .9. I randomly select one of the coins, each with probability .5, and hand it to you. You flip the coin 4 times and the result is $HHHH$. What is the probability that I gave you the biased coin?

Written using probability notation, the problem is as follows:

$$P(\text{heads}|\text{fair}) = .5 \tag{12}$$
$$P(\text{heads}|\text{bias}) = .9 \tag{13}$$
$$P(\text{bias}) = .5 \tag{14}$$

Find $P(\text{bias}|HHHH)$.

We can do this calculation with Bayes rule:

$$P(\text{bias}|HHHH) = P(HHHH|\text{bias})\frac{P(\text{bias})}{P(HHHH)} \tag{15}$$
$$\tag{16}$$

We need to calculate three terms on the right-hand side of 15.

1. $P(HHHH|\text{bias}) = .9^2 = .6561$ (four independent coin flips, each has probability .9 of landing heads.
2. $P(\text{bias}) = .5$ (given)
3. $P(HHHH)$ is the hardest term to calculate.

$$\begin{aligned} P(HHHH) &= P(HHHH \cap \text{fair}) + P(HHHH \cap \text{bias}) && (17)\\ &= P(HHHH|\text{fair})P(\text{fair}) + P(HHHH|\text{bias})P(\text{bias}) && (18)\\ &= .5^4 \cdot .5 + .9^4 \cdot .5 && (19)\\ &= .3593 && (20) \end{aligned}$$

Finally,

$$\begin{aligned} P(\text{bias}|HHHH) &= P(HHHH|\text{bias})\frac{P(\text{bias})}{P(HHHH)} && (21)\\ &= .5451 \cdot \frac{.5}{.3593} && (22)\\ &\approx .913 && (23) \end{aligned}$$

Alternatively, we can do this calculation with odds

$$\begin{aligned} \text{odds}(\text{bias}|HHHH) &= \text{odds}(\text{bias}) \cdot \frac{P(HHHH|\text{bias})}{P(HHHH|{\sim}\text{bias})} && (24)\\ &= 1 \cdot \frac{.9^4}{.5^4} && (25)\\ &\approx 10.5 && (26) \end{aligned}$$

Then we just go back from probability to odds: $P = \frac{\text{odds}}{1+\text{odds}} \approx .913$

3 Random variables

Suppose a certain experiment has N outcomes, with probabilities $p_1, p_2, \ldots, p_N$. A random variable X takes a particular value for each outcome of the experiment. Let's call the outcomes $x_1, x_2, \ldots, x_N$.

Example: Six-sided die has six outcomes: $x_1 = 1, x_2 = 2, \ldots, x_6 = 6$ with $p_1 = p_2 = \cdots = p_6 = 1/6$.

Two random variables X and Y are defined to be *independent*

$$P(X = x \cap Y = y) = p(X = x) \cdot P(y = y) \tag{27}$$

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4 Expected value

Expected value of a random variable is its average value over all possible outcomes, weighted by the outcomes' probabilities.

$$E[X] = \sum_{i=1}^{N} x_i p(X = x_i) \tag{28}$$

5 Expected value of a sum

A jar has three red balls and two blue balls.

1. Draw one ball. What's the expected number of red balls?
2. Draw two balls *with* replacement. What's the expected number of red balls?
3. Draw two balls *without* replacement. What's the expected number of red balls?

In problem (1), we are calculating $E[X]$, where

$$X = \begin{cases} 0 & \text{if ball is red,} \\ 1 & \text{if ball is blue} \end{cases} \tag{29}$$

Since $P(\text{red}) = 3/5$, $E[X] = 2/5 \cdot 0 + 3/5 \cdot 1 = 3/5$

Now let's try problem 2. We could solve the problem directly, calculating $E[Y]$, where

$$Y = \begin{cases} 0 & \text{red, red,} \\ 1 & \text{red, blue,} \\ 1 & \text{blue, red,} \\ 2 & \text{blue, blue} \end{cases} \tag{30}$$

As you can verify by calculating the probabilities of these four outcomes, $E[Y] = 6/5$. There's an easier way of solving the problem. Write Y as the sum of two random variables X_1 and X_2, where

$$X_1 = \begin{cases} 0 & \text{first ball is red,} \\ 1 & \text{first ball is blue} \end{cases} \tag{31}$$

$$X_2 = \begin{cases} 0 & \text{second ball is red,} \\ 1 & \text{second ball is blue} \end{cases} \tag{32}$$

Since X_1 and X_2 are independent, you might guess that we can calculate their expectations separately.

$$E[X_1 + X_2] = E[X_1] + E[X_2] \tag{33}$$

$$= 3/5 + 3/5 \tag{34}$$

$$= 6/5 \tag{35}$$

Now try problem 3. You can verify that the answer is 6/5 again. It turns out that the relationship $E[X_1 + X_2] = E[X_1] + E[X_2]$ holds for any two random variables X_1 and X_2, even if X_1 and X_2 are not independent.

6 Expected time

We talked about a few problems where you have to calculate the expected number of trials until some event happens. Here's one example: Flip a coin until it lands on heads. Let k be the total number of coin flips. (For example, if the result of the first flip is heads, then $k = 1$.) What's $E[k]$?

There are at least two ways to solve it.

Method 1: direct.

Suppose $k = 5$. Then the first five flips must have been $TTTTH$. This sequence has probability $\frac{1}{2^5}$.

$$E[k] = P(k=1)\cdot 1 + P(k=2)\cdot 2 + P(k=3)\cdot 3 + \dots \tag{36}$$
$$= \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \dots \tag{37}$$

This is an infinite series. However, we can calculate it using the fact that $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1$. We write the sum as follows

$$E[k] = \frac{1}{2^1} + \quad \frac{1}{2^2} + \quad \frac{1}{2^3} + \dots \tag{38}$$
$$+ \quad \frac{1}{2^2} + \quad \frac{1}{2^3} + \dots \tag{39}$$
$$+ \quad \frac{1}{2^3} + \dots \tag{40}$$

Now we sum along the rows, giving $1, \frac{1}{2}, \frac{1}{4}$, etc. Adding these, we get $E[k] = 2$.

Method 2: recursive.

$$E[k] = P(\text{first flip is heads}) \cdot E[k|\text{first flip is heads}] + P(\text{first flip is tails}) \cdot E[k|\text{first flip is tails}] \tag{41}$$
$$= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot E[k|\text{first flip is tails}] \tag{42}$$

$$E[k|\text{first flip is tails}] = 1 + E[\text{number of flips following first tails}] \tag{43}$$

But after that first tails, we're starting the game all over again, so

$$E[\text{number of flips following first tails}] = E[k] \tag{44}$$

So we have

$$E[k] = \frac{1}{2} \cdot 1 + \frac{1}{2}(1 + E[k]) \tag{45}$$

Solving, we get $E[k] = 2$

You can generalize this argument to solve the following problem: in a given trial, event E occurs with probabilty p. What's the expected number of trials until E occurs (including the trial where E occurs)?

Let k be the number of trials.

$$E[k] = p \cdot 1 + (1-p) \cdot (1 + E[k]) \tag{46}$$
$$E[k] = \frac{1}{p} \tag{47}$$

7 Challenge problems

1. Use equation 6 to write $P(A \cap B \cap C)$ in terms of $P(A)$, $P(B|A)$, and $P(C|A, B)$.
2. (i) Find $P(A)$ in terms of odds(A). (ii) Show that $P(A) > P(B)$ implies odds$(A) >$ odds(B).
3. With probability 1/2, I give you a 52-card deck without jokers. With probability 1/2, I give you a 54-card deck with two jokers. You look at the top 27 cards (but don't count the number of cards in the deck!) Given that none of them are jokers, what's the probability that I gave you the deck with jokers?
4. Keep rolling a die until all six sides have shown at least once. What's the expected number of rolls? [Hint: what's the expected number of rolls until the first new side? second new side?]
5. There are $n!$ permutations of the list $1, 2, 3, \dots, n$. A permutation may have a number of *fixed points*, i.e., elements of the list that are unchanged by the permutation. For example, the permutation $1, 2, 3, 4, 5 \rightarrow 1, 3, 4, 2, 5$ has two fixed points (the first and last elements). What's the expected number of fixed points of a randomly chosen permutation? [Hint: there's an elegant solution using the sum property $E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$. Just figure out the right X_i.]