INTERMEDIATE BAMO PREPARATION

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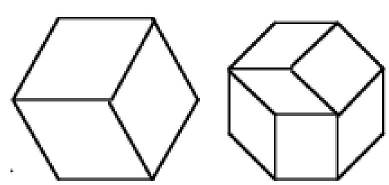
This handout is intended to serve as preparation for the BAMO-8. I've intentionally put way more problems on here than time allows in the hopes that this will be useful to you after the session is over, so dont feel bad if you don't finish everything!
From 1999-2007, there was only one BAMO exam. Since 2008 there have been two, the BAMO-8 and the BAMO-12. The BAMO-8 compares in difficulty to the first three questions from old exams, so this handout consists mainly of these problems. I've loosely grouped the questions together in categories and sorted them by difficulty within each category, but of course these are by no means exhaustive: anything and everything can appear on the BAMO...

1. Show You Can

1 (BAMO 2009.1)**.** Consider a 7×7 chessboard that starts out with all the squares white. We start painting squares black, one at a time, according to the rule that after painting the first square, each newly painted square must be adjacent along a side to only the square just previously painted. The final painted figure will be a connected "snake" of squares.

(1) Show that it is possible to paint 31 squares.
(2) Show that it is possible to paint 32 squares.
(3) Show that it is possible to paint 33 squares.

2 (BAMO 2002.2)**.** In the illustration, a regular hexagon and a regular octagon have been tiled with rhombuses. In each case, the sides of the rhombuses are the same length as the sides of the regular polygon.



(1) Tile a regular decagon (10-gon) into rhombuses in this manner.
(2) Tile a regular dodecagon (12-gon) into rhombuses in this manner.
(3) How many rhombuses are in the tiling by rhomubses of a 2002-gon?

3 (BAMO 2001.1)**.** Each vertex of a regular 17-gon is colored red, blue or green in such a way that no two adjacent vertices have the same color. Call a triangle "multicolored"

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if its vertices are colored red, blue, and green, in some order. Prove that the 17-gon can be cut along nonintersecting diagonals to form at least two multicolored triangles.
(A *diagonal* of a polygon is a line segment connecting two nonadjacent vertices. Diagonals are called *nonintersecting* if each pair of them either inersects in a vertex or do not intersect at all.)

4 (BAMO 2005.3)**.** Let n be an integer greater than 12. Points $P_1, P_2, \ldots, P_n, Q$ in the plane are distinct. Prove that for some i, at least $n/6 - 1$ of the distances

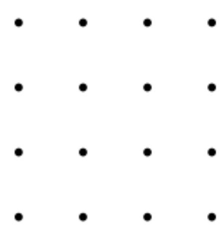
$$P_1P_i, P_2P_i, \ldots, P_{i-1}P_i, P_{i+1}P_i, \ldots, P_nP_i$$

are less than P_iQ.

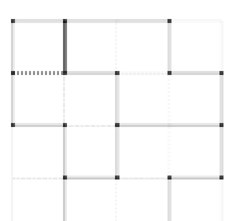
5 (Putnam 2002 A2)**.** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

2. Show You Can't

6 (2009.1)**.** A square grid of 16 dots contains the corners of nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of 14 squares whose sides are parallel to the sides of the grid. What is the smallest possible number of dots you can remove so that, after removing those dots, each of the 14 squares is missing at least one corner?
Justify your answer by showing both that the number of dots you claim is sufficient and by explaining why no smaller number of dots will work.



7 (BAMO 2008.4)**.** Determine the greatest number of figures congruent to  that can be placed in a 9×9 grid (without overlapping), such that each figure covers exactly 4 unit squares. The figures can be rotated and flipped over. For example, the picture below shows that at least 3 such figures can be placed in a 4×4 grid.



8 (BAMO 2004.1)**.** A *tiling* of the plane with polygons consists of placing the polygons in the plane so that the interiors of polygons do not overlap, each vertex of one polygon

coincides with a vertex of another polygon, and no point of the plane is left uncovered. A *unit* polygon is a polygon with all sides of length one.
It is quite easy to tile the plane with infinitely many unit squares. Likewise, it is easy to tile the plane with infinitely many unit equilateral triangles.

(1) Prove that there is a tiling of the plane with infinitely many unit squares and infinitely many unit equilateral triangles in the same tiling.
(2) Prove that it is impossible to find a tiling of the plane with infinitely many unit squares and finitely many (and at least one) unit equilateral triangles in the same tiling.

9 (BAMO 2007.2)**.** The points of the plane are colored in black and white so that whenever three vertices of a parallelogram are the same color, the fourth vertex is that color, too. Prove that all the points of the plane are the same color.

10. Suppose there is a natural number placed at each lattice point in the xy-plane, such that every number is equal to the average of its four neighbors. (For example, the number placed at the origin $(0, 0)$ is equal to the average of the numbers at $(-1, 0)$, $(1, 0)$, $(0, 1)$, and $(0, -1)$). Show that all the numbers are equal. Must this still be true if you replace "natural number" with "integer" in the problem statement? (A *lattice point* is a point whose coordinates are integers).

3. Numbers as Sums

11 (BAMO 1999.1)**.** Prove that among any 12 consecutive positive integers there is at least one which is smaller than the sum of its proper divisors. (The proper divisors of a positive integer n are all positive integers other than 1 and n which divide n. For example, the proper divisors of 14 are 2 and 7).

12 (BAMO 2003.1)**.** An integer is a *perfect* number if and only if it is equal to the sum of all of its divisors except itself. For example, 28 is a perfect number since $28 = 1 + 2 + 4 + 7 + 14$.
Let $n!$ denote the product $1 \cdot 2 \cdot 3 \cdots n$, where n is a positive integer. An integer is a *factorial* if and only if it is equal to $n!$ for some positive integer n. For example, 24 is a factorial number since $24 = 4! = 1 \cdot 2 \cdot 3 \cdot 4$.
Find all perfect numbers greater than 1 that are also factorials.

13 (BAMO 2000.1)**.** Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1. (Two integers are relatively prime if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus $37 = 22 + 15$ represents the number 37 in the desired way).

14 (BAMO 2005.1)**.** An integer is called *formidable* if it can be written as a sum of distinct powers of 4, and *successful* if it can be written as a sum of distinct powers of 6. (We are only allowing nonnegative integer powers). Can 2011 be written as a sum of a formidable number and a successful number? Prove your answer.

15 (BAMO 2006.2)**.** Since $24 = 3 + 5 + 7 + 9$, the number 24 can be written as the sum of at least two consecutive odd positive integers.

(1) Can 2005 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If not, provide a proof why not.
(2) Can 2006 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how this can be done. If no, provide a proof why not.

4. Geometry

16 (BAMO 1999.2)**.** Let C be a circle in the xy-plane with center on the y-axis and passing through $A = (0, a)$ and $B = (0, b)$ with $0 < a < b$. Let P be any other point on the circle and Q the intersection of the line through P and A with the x-axis, and let $O = (0, 0)$. Prove that $\angle QBP = \angle BOP$.

17 (BAMO 2002.1)**.** Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to triangle ABC. If M is the center of this square, find the measure of $\angle MBC$.

18 (BAMO 2001.2)**.** Let $JHIZ$ be a rectangle, and let A and C be points on sides ZI and ZJ, respectively. The perpendicular from A to CH intersects line HI in X, and the perpendicular from C to AH intersects line HJ in Y. Prove that X, Y and Z are collinear (lie on the same line)

19 (BAMO 2004.2)**.** A given line passes through the center O of a circle. The line intersects the circle at points A and B. Point P lies in the exterior of the circle and does not lie on the line AB. Using only an unmarked straightedge, construct a line through P, perpendicular to the line AB. Give complete instructions for the construction and prove that it works.

5. Miscellaneous

20 (BAMO 2006.1)**.** All the chairs in a classroom are arranged in a square $n \times n$ array (in other words, n columns and n rows) and every chair is occupied by a student. The teacher decides to rearrange students according to the following rules:

- Every student must move to a new chair.
- A student can only move ot an adjacent chair in teh same row or to an adjacent chair in the same column. In other words, each student can move only one chair horizontally or vertically.

(Note that the rules allow two students in adjacent chairs to exchange places). Show that this procedure can be done if n is even, and cannot be done if n is odd.

21 (BAMO 2009.2)**.** The *Fibonacci sequence* is the list of numbers that begins $1, 2, 3, 5, 8, 13$ and continues with each subsequent number being the sum of the pervious two.
Prove that for every positive integer n, when the first n elements of the Fibonacci sequence are alternately added and substracted, the result is an element of the sequence or the negative of an element of the sequence. For example, when $n = 4$ we have

$$1 - 2 + 3 - 5 = -3$$

and 3 is an element of the Fibonacci sequence.