GAUSSIAN INTEGERS

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Exercise 1. There is a number below 65 which is a sum of two squares in two different ways: $a^2 + b^2 = c^2 + d^2$ where a, b, c, d are positive integers and $\{a, b\} \neq \{c, d\}$. What is the number? (Hint: Make a table of small squares and add them together two at a time.)

Exercise 2. For the following Gaussian integers α and β, decide if α is a factor of β.
a) $\alpha = 1 + i$, $\beta = 1 + 3i$.
b) $\alpha = 1 + 2i$, $\beta = 3 + 4i$.
c) $\alpha = 2 + 3i$, $\beta = 14 - 5i$.

Exercise 3. a) Use norms to explain why $2 + 5i$ is not a factor of $1 + 6i$ and why $1 + 2i$ is not a factor of $7 + 3i$.
b) Can we use norms to explain any of the results of exercise 2?

Call a nonzero Gaussian integer α *prime* if it isn't ± 1 or $\pm i$ and its only factorizations are $\alpha \cdot 1$, $(-\alpha)(-1)$, $(i\alpha)(-i)$, and $(-i\alpha)i$.

Exercise 4. If α is a Gaussian integer and $\mathrm{N}(\alpha) = p$ is a prime number, then show α is a prime in $\mathbf{Z}[i]$.

Exercise 5. In $\mathbf{Z}[i]$, consider the factorizations $10 = 2 \cdot 5$ and $10 = (1 + 3i)(1 - 3i)$. Show neither of these is a prime factorization of 10 in $\mathbf{Z}[i]$ and break up the factors in each case further until you reach a factorization into primes in $\mathbf{Z}[i]$. Can you match up the two prime factorizations of 10 which you get in this way?

Exercise 6. The equations $5 = (2 + i)(2 - i)$ and $5 = (1 + 2i)(1 - 2i)$ both express 5 as a product of primes in $\mathbf{Z}[i]$. Can you match up the terms in these two prime factorizations of 5? (Hint: Why don't the two equations $6 = 2 \cdot 3$ and $6 = (-2)(-3)$ violate unique prime factorization in $\mathbf{Z}$?)

Exercise 7. Use each of the two equations $65 = 1^2 + 8^2$ and $65 = 4^2 + 7^2$ to factor 65 into primes in $\mathbf{Z}[i]$. (Hint: Think about a sum of two squares as a norm: $a^2 + b^2 = \mathrm{N}(a + bi) = (a + bi)(a - bi)$.) Compare the prime factors of 65 which you get by each method.

Exercise 8. Use the ideas from Exercise 7 in reverse to find an integer larger than 65 that is a sum of two nonzero squares in at least two ways.

Exericse 9. Find an integer that is a sum of two nonzero squares in at least three ways.

Exericse 10. Factor $73 + 14i$ into primes in $\mathbf{Z}[i]$. (Hint: Its norm is $5525 = 5^2 \cdot 13 \cdot 17$.)