Numbers and Sequences part 1: Polynomials and Pascal

Joshua Zucker, Berkeley Math Circle, April 12, 2011

Problem 1. Into how many pieces can you divide a circular pizza with 1 straight cut? With 2? With 3? With 4? We'll assume your cuts must go all the way across, that you cannot rearrange the pieces in between cuts, and that you want to make the most pieces possible.

Problem 2. Into how many pieces can you divide a circular pizza by labeling 2 points on the edge and then connecting them? With 3 points, connected in all possible ways? 4 points? 5 points? 6? Does it depend on how you arrange the points?

Introduce the method of finite differences, and the notation.

Problem 3. Given the sequence a_n = 1, 4, 7, 10, 13, …, I hope you can all see that the differences Da_n = 3, 3, 3, 3, …. That helps you calculate the next term, so do that, and then also use it to find a general formula for a_n in terms of n.

Problem 4. If the first term of a sequence is 7, and it has first difference a constant 2, write a formula for the sequence.

Problem 5. How about a formula for the sequence 2, 9, 22, 41, 66, 97, 134, …?

Problem 6. If the first term of a sequence is 11, the first term of its first differences is 5, and the second difference is a constant 12, write a formula for the sequence.

Problem 7. What about 0, 6, 24, 60, 120, 210, 336, …? This one might be amenable to a good guess-and-check, too, as well as the finite differences method.

Problem 8. Now let's work the other way: if $a_n = mx + b$, what will Da_n be equal to? What if a_n has a quadratic pattern, maybe $Ax^2 + Bx + C$? Cubic?

Problem 9. If the first term of a sequence is w, the first term of its first differences is x, the first term of its second differences is y, and the third difference is a constant z, write a formula for the sequence.

Problem 10. Instead of powers of x like x^2 and x^3, the **falling** powers are very useful here. We define them as $x^{\underline{2}} = x\,(x-1)$ and $x^{\underline{3}} = x\,(x-1)\,(x-2)$ and so on. Explain why these are convenient.

Problem 11. Now relate what you've learned so far to Pascal's Triangle.

Problem 12. What does all the above stuff about differences tell you about the sum of a sequence? Can you find a formula for the sum of the numbers 1 through n? How about for the sum of the squares? Cubes? Fourth powers?

Problem 13. Make a nice formula for 1, 2, 4, 8, 16, 31, 57, 99, 163, 256, 386, 562, 794, 1093, 1471, 1941, 2517, 3214, 4048, …

Problem 14. What happens when you try this same technique on 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, …?

Problem 15. What about if you try it on 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …?