Piecewise Linear Functions
BMC Advanced, November 2, 2010
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Problems

1. Show that

   a) An equation of a straight line in the plane is either $y = mx + b$ or $x = a$, where m, b, and a are real numbers.
   b) The graph of an equation $y = mx + b$ is a straight line in the plane.
   c) A straight line ℓ in the plane divides the plane into two regions such that (i) for any two points A, B lying in the same region, the line segment AB does not intersect ℓ, (ii) for any two points A, B lying in different regions, the line segment AB intersects ℓ at precisely one point.
   d) For $A = (a, b)$, $B = (c, d)$:
   $$AB = \{(x, y) \in \mathbb{R}^2 : x = (1-t)a + tc,\ y = (1-t)b + td,\ 0 \le t \le 1\}.$$

2. Let $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$, where x, y are real numbers. Show that
   $$x \wedge y = y \wedge x, \quad x \vee y = y \vee x,$$
   $$x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee y \vee z) = (x \vee y) \vee z$$
   $$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

3. Show that any lattice polynomial in variables x_1, x_2 is equivalent to $x_1 \wedge x_2$ or to $x_1 \vee x_2$, that is,
   $$p(x_1, x_2) = x_1 \wedge x_2 \quad \text{or} \quad p(x_1, x_2) = x_1 \vee x_2$$
   for all real numbers x_1, x_2.

4. Let p be a lattice polynomial in variables $x_1, \ldots, x_n$. Show that for any set of numbers $\{a_1, \ldots, a_n\}$ the number $p(a_1, \ldots, a_n)$ belongs to the set $\{a_1, \ldots, a_n\}$. (Hint: use induction on weight.)

5*. Let $g_1, \ldots, g_n$ be linear functions on an interval I, and let p be a lattice polynomial in n variables. Show that
   $$f(x) = p(g_1(x), \ldots, g_n(x))$$
   is a PL-function on I.

6*. Let p be a lattice polynomial in variables $x_1, \ldots, x_n$. Show that there is a finite family of sets $\{S_j\}_{j \in J}$, $S_j \subseteq \{1, \ldots, n\}$, such that
   $$p(x_1, \ldots, x_n) = \bigvee_{j \in J} \bigwedge_{k \in S_j} x_k,$$
   where $x_1, \ldots, x_n$ are real numbers.

Problems marked by * are more difficult than other problems.