Problem Solving Using Prime Factorization

Definition 1 A natural number greater than 1 is said to be **prime** if its only natural number divisors are 1 and itself. Natural numbers greater than 1 that are not prime are **composite**.

Theorem 1 The Fundamental Theorem of Arithmetic. Every natural number, other than 1, can be factored into a product of primes in only one way, apart from the order of the factors.

1. Find positive integers x and y that satisfy both
$$xy = 40 \text{ and } 31 = 2x + 3y.$$
2. (1998 AHSME #6) Suppose that 1998 is written as a product of two positive integers whose difference is as small as possible. What is this difference?
3. (2005 AMC 10A #15) How many positive cubes divide $3! \cdot 5! \cdot 7!$?
4. (2001 AMC 12 #21) The product of four positive integers a, b, c, d is $8!$ and they satisfy the equations
$$\begin{aligned} ab + a + b &= 524 \\ bc + b + c &= 146 \\ cd + c + d &= 104. \end{aligned}$$
What is $a - d$?
5. Find the smallest positive integer n such that $n/2$ is a perfect square, $n/3$ is a perfect cube, and $n/5$ is a perfect fifth power.
6. Show that $\log_{10} 2$ is irrational.
7. Find all positive integers n such that $2^8 + 2^{11} + 2^n$ is a perfect square.
8. (1999 AHSME #6) What is the sum of the digits of the decimal form of the product $2^{2004} \cdot 5^{2006}$?
9. (2002 AMC 10B #14) The number $25^{64} \cdot 64^{25}$ is the square of a positive integer N. What is the sum of the digits of N?
10. (2002 AMC 10A #14 and 12A #12) Both roots of the quadratic equation
$$x^2 - 63x + k = 0$$
are prime numbers. What is the number of possible values of k?
11. (1986 AHSME #23) Let
$$N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1.$$
How many positive integers are factors of N?

12. (2003 AMC 12A #23) How many perfect squares are divisors of the product

$$1! \cdot 2! \cdot 3! \cdots 9!?$$

13. (1990 AHSME #11) How many positive integers less than 50 have an odd number of positive integer divisors?

14. (1993 AHSME #15) For how many values of n will an n-sided regular polygon have interior angles with integer degree measures?

15. (2002 AMC 12 #20) Suppose that a and b are digits, not both nine and not both zero, and the repeating decimal

$$0.abababab\cdots$$

is expressed as a fraction in lowest terms. How many different denominators are possible?

16. (1996 AHSME #29) Suppose that n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors. How many positive divisors does $6n$ have?

17. Find the smallest number with 28 divisors.

18. Given distinct integers a, b, c, d such that

$$(x-a)(x-b)(x-c)(x-d)-4=0$$

has an integral root r, show that $4r = a+b+c+d$.

19. Given positive integers a, b, c, d such that $a^3 = b^2$, $c^3 = d^2$, and $c - a = 25$, determine a, b, c, d.

20. Determine all positive rational solutions of $x^y = y^x$.