Plane Geometry and a Recent BAMO Brilliancy Solution

Berkeley Math Circle

by Zvezdelina Stankova
Berkeley Math Circle Director

May 3, 2011

1. Warm-Ups

Problem 0!. (Distances to Inscribed Equilateral Triangle) An equilateral triangle is inscribed in a circle. An arbitrary point is selected on the circle. The distances from this point to two of the vertices of the triangle are 4 cm and 7 cm, respectively. What could the distance from the point to the third vertex be? Give all possible answers and prove your claim.

Problem 2. (Distances in an Inscribed Quadrilateral) A quadrilateral is inscribed in a circle. The two diagonals of the quadrilateral are 9 cm and 6 cm long. Three of the sides of the quadrilateral, going clockwise around the circle, are 8 cm, 3 cm, and 3 cm long. How long is the fourth side of the quadrilateral? Give all possible answers and prove your claim.

Problem 3. (Medians and Angle Bisectors) Let CM be a median in $\triangle ABC$, and CP – a median in $\triangle AMC$. Given that $AB = 2AC = 2CM$, prove that CM is the angle bisector of $\angle PCB$.

Problem 2^2. (Altitudes and Circumcircle) $\triangle ABC$ is inscribed in circle k. The extensions of the two altitudes AE and BD of $\triangle ABC$ intersect k in points A_1 and B_1, respectively. If $\angle C = 60°$, prove that $AA_1 = BB_1$. Is the converse true?

Problem 5. (Napoleon's Theorem) Externally to $\triangle ABC$ are drawn three equilateral triangles ABP, BCQ, and CAN, as well as the circles described about each of these equilateral triangles.

(a) Prove that the three circles intersect in a point.
(b) Prove that the lines AQ, BN, and CP also intersect in a point.
(c) Prove that the centers of the three circles form an equilateral triangle.

2. BAMO Geometry Jewels

Problem 3!. (BAMO '06) In $\triangle ABC$, three points A_1, B_1 and C_1 are selected on sides BC, CA, and AC, respectively, so that the segments AA_1, BB_1 and CC_1 intersect in some point P. Prove that P is the centroid of $\triangle ABC$ if and only if P is the centroid of $\triangle A_1B_1C_1$.

Problem 7. (BAMO '07) In $\triangle ABC$, D and E are two points inside side BC such that $BD = CE$ and $\angle BAD = \angle CAE$. Prove that $\triangle ABC$ is isosceles.

Problem 2^3. (BAMO '10) Acute $\triangle ABC$ has $\angle BAC < 45°$. Point D lies in the interior of $\triangle ABC$ so that $BD = CD$ and $\angle BDC = 4\angle BAC$. Point E is the reflection of C across line AB, and point F is the reflection of B across line AC. Prove that lines AD and EF are perpendicular.

Problem 3^2. (BAMO '11) Three circles k_1, k_2, and k_3 intersect in point O. Let A, B, and C be the second intersection points (other than O) of k_2 and k_3, k_1 and k_3, and k_1 and k_2, respectively. Assume that O lies inside of the triangle ABC. Let lines AO, BO, and CO intersect circle k_1, k_2, and k_3 for a second time at points A', B', and C', respectively. If $|XY|$ denotes the length of segment XY, prove that

$$\frac{|AO|}{|AA'|} + \frac{|BO|}{|BB'|} + \frac{|CO|}{|CC'|} = 1.$$