Plane Geometry II: Geometry Problems on the Circle

Berkeley Math Circle – Beginners

by Zvezdelina Stankova

Berkeley Math Circle Director

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Note: We shall work on the problems below over several circle sessions. Bring this handout with you to all Geometry II sessions of BMC–beginners. Try to understand what the overarching problems say and draw pictures for them as best as you can. You are not expected to be able to solve the problems on your own, at least not in the beginning of the Geometry II series at BMC–beginners. Some of the problems in this handout are from *"Mathematical Olympiads"*, part II by Stoyan Budurov and Dimo Serafimov, State Publishing Company "Narodna Prosveta", Sofia, 1985.

To fully prepare for this geometry module,

(a) Review the *BMC-beginners Geometry I handout* from September '10. Specifically, we will need here:
- the criteria for similar and congruent triangles;
- the concepts of the circumcircle of a triangle, and the perpendicular bisector of a segment;
- the concepts of medians, altitudes, and angle bisectors in a triangle.

(b) Read Session 11 on *Circle Geometry* by Ivan Matić from *"A Decade of the Berkeley Math Circle – the American Experience"*, volume I, edited by Zvezdelina Stankova and Tom Rike, published by AMS, 2009, www.ams.org or www.amazon.com. Specifically, we will need here:
- the concepts of inscribed and central angles in a circle, and the theorem about their measures.
- the concept of an inscribed trapezoid and what is known about its diagonals.

For on-going study of plane geometry:

(c) Read *Geometry (part I): Planimetry* by Kiselev, 2006, www.sumisdat.org.

1. Overarching Problems

(1) **(Distances to Inscribed Equilateral Triangle)** An equilateral triangle is inscribed in a circle. An arbitrary point is selected on the circle. The distances from this point to two of the vertices of the triangle are 4 cm and 7 cm, respectively. What could the distance from the point to the third vertex be? Give all possible answers and prove your claim.

(2) **(Distances in an Inscribed Quadrilateral)** A quadrilateral is inscribed in a circle. The two diagonals of the quadrilateral are 9 cm and 6 cm long. Three of the sides of the quadrilateral, going clockwise around the circle, are 8 cm, 3 cm, and 3 cm long. How long is the fourth side of the quadrilateral? Give all possible answers and prove your claim.

A *median* in a triangle is the segment connecting a vertex of the triangle with the midpoint of the opposite side. An *angle bisector* is a ray passing through the vertex of an angle and splitting the angle into two equal parts. An *altitude* in a triangle is a segment from a vertex of the triangle to the opposite side so that the segment is perpendicular to that side.

(3) **(Medians and Angle Bisectors)** Let CM be a median in $\triangle ABC$, and CP – a median in $\triangle AMC$. Given that $AB = 2AC = 2CM$, prove that CM is the angle bisector of $\angle PCB$.

(4) **(Four Equal Angles)** The altitude, the angle bisector, and the median drawn from the same vertex of a triangle divide the angle at that vertex into four equal parts. Find the measures angles of the angles of the triangle and prove your claim.

(5) **(Altitudes and Circumcircle)** $\triangle ABC$ is inscribed in circle k. The extensions of the two altitudes AE and BD of $\triangle ABC$ intersect k in points A_1 and B_1, respectively. If $\angle C = 60°$, prove that $AA_1 = BB_1$. Is the converse true?

(6) **(Napoleon's Theorem)** Externally to $\triangle ABC$ are drawn three equilateral triangles ABP, BCQ, and CAN, as well as the circles described about each of these equilateral triangles.

(a) Prove that the three circles intersect in a point.

(b) Prove that the lines AQ, BN, and CP also intersect in a point.

(c)* Prove that the centers of the three circles form an equilateral triangle.

2. Warm-Ups on Cyclicity and Parallelism

A *cyclic* figure is a figure *inscribed in a circle.*

(7) **(Cyclic Triangles)** Is any *equilateral* triangle is cyclic? Why? Is *any* triangle cyclic? If yes, is it inscribed in exactly one circle, or is it possible to describe two different circles about the same triangle? Why? Explain and formulate a theorem.

(8) **(Cyclic Quadrilaterals)** Is any *quadrilateral* cyclic? Why or why not? Can you formulate a theorem that states precisely which quadrilaterals are cyclic? Explain.

(9) **(Cyclic Trapezoids)** Is any *trapezoid* cyclic? Is any cyclic trapezoid *isosceles*? Is any isosceles trapezoid cyclic? What do you know about the *diagonals* of an isosceles trapezoid? of a cyclic trapezoid? Formulate a theorem that states precisely which trapezoids are cyclic.

Two lines l and m in the plane are *parallel* if they do not intersect. A *transversal* t to lines l and m is simply a line that intersects both l and m, say, t intersects l in point X and m in point Y. If A lies on l and D lies on m so that A and D are on opposite sides of t, the two angles $\angle AXY$ and $\angle DYX$ are called *interior alternating angles.*

(10) **(Parallel Lines Theorem)** Two lines are parallel if and only if the interior alternating angles formed by some transversal are equal.

3. Truths on the Circle

Given segment AB, its *perpendicular bisector* is a line l which passes through the midpoint of AB and is perpendicular to AB.

(11) **(Perpendicular Bisector Theorem)** The *locus of points* X in the plane that are equidistant from A and B (i.e., $XA = XB$) is the perpendicular bisector of AB.

(12) **(Circumcircle Theorem)** Any triangle $\triangle ABC$ is inscribed in precisely one circle k, called the *circumcircle* of the triangle. The center of k is the intersection is the three perpendicular bisectors of the sides of $\triangle ABC$. This center is called the *circumcenter* of $\triangle ABC$.

If points A, B, and C lie on circle k, then $\angle ABC$ is said to be an *inscribed* angle in k. One of the arcs $\overset{\frown}{AC}$ on k does **not** contain point B; we say that this arc $\overset{\frown}{AC}$ is *subtended* by inscribed $\angle ABC$. If O is center of k, the angle $\angle AOC$ is called the *central* angle corresponding to inscribed $\angle ABC$. Note that both the inscribed angle ABC and the central angle AOC subtend the same arc $\overset{\frown}{AC}$.

(13) **(Inscribed Angle Theorem)** An inscribed angle is half of its corresponding central angle. Two inscribed angles subtending the same arc in a circle are equal.

(14) **(Diameter Theorem)** Let $\angle ABC$ be an inscribed angle in circle k. Then AB is a diameter of k if and only if $\angle ABC$ is right. Equivalently, a point M on side AB of $\triangle ABC$ is the circumcenter of $\triangle ABC$ if and only if $\angle ABC$ is right.

(15) **(Chord Theorem)** A *chord* in a circle is a segment connecting two points on the circle. Two chords in circle k are equal if and only if two of the arcs that they determine are equal.

(16) **(Cyclic Quadrilateral Theorem)** Consider line AB and two points C and D not on it.
  (a) If C and D lie on the same side of line AB, then quadrilateral $ABCD$ is cyclic if and only if $\angle ACB = \angle ADB$.
  (b) If points C and D lie on opposites sides of line AB, then quadrilateral $ABCD$ is cyclic if and only if $\angle ACB + \angle ADB = 180°$.

(17) **(Cyclic Trapezoids)** For a trapezoid, the following conditions are equivalent:
  (a) the trapezoid is cyclic;
  (b) the trapezoid is isosceles;
  (c) the trapezoid has diagonals equal in length.

4. Truths about Triangles

(18) **(Similar Triangles Theorem)** Two triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are similar if one of the following is satisfied:
  (a) (RRR) The three ratios of the sides in one triangle to the corresponding sides in the other triangle are all equal. For example, $AB/A_1B_1 = BC/B_1C_1 = CA/C_1A_1$.
  (b) (AA) Two angles in one triangle are correspondingly equal to two angles in the other triangle. For example, $\angle BAC = \angle B_1A_1C_1$ and $\angle ABC = \angle A_1B_1C_1$.
  (c) (RAR) For two sides and an angle between them in one triangle and analogous three elements in the other triangle, the two angles are equal, and the two ratios of the corresponding sides in the two triangles are equal. For example, $\angle BAC = \angle B_1A_1C_1$ and $BA/B_1A_1 = CA/C_1A_1$.

5. Homework 1

(19) **(Inscribing an Equilateral Triangle)** Mark a point O in the plane. Draw a circle k with center O and some radius r. Mark a point A_1 on k. With center A_1, draw an arc with the same radius r, until it intersects k in point A_2. With center A_2, draw an arc with the same radius r , until it intersects k in some other point A_3. Continue in the same manner to construct points A_4, A_5, and A_6. Explain why the hexagon $A_1A_2A_3A_4A_5A_6$ is regular, and why $\triangle A_1A_3A_5$ and $\triangle A_2A_4A_6$ are both equilateral triangles (inscribed in k). Think of other ways to create an equilateral triangle inscribed in a given circle k.

(20) **(Drawing a Perpendicular Bisector)** Let AB be a segment. Using a compass and a straightedge only (no markings on the straightedge), we will construct the perpendicular bisector of AB as follows. Draw a circle k_1 with center A and an arbitrary radius r, as long as $r > AB$. Draw a circle k_2 with center B and the same radius r. Let k_1 and k_2 intersect in points M and N. Explain why MN is the perpendicular bisector of AB.

(21) **(Cyclic or Not Cyclic?)** Reason why or why not the following figures are cyclic:
  (a) An equilateral triangle; an isosceles triangle; a scalene triangle; an arbitrary triangle.
  (b) A square; a rectangle.
  (c) A trapezoid; an isosceles trapezoid.
  (d) An arbitrary quadrilateral.

(22) **(Drawing the Circumcircle)** Given $\triangle ABC$, draw the perpendicular bisectors of side AB and side BC, and let them intersect in point O. Explain why O is equidistant from A, B, and C (i.e., why $OA = OB = OC$), and hence conclude that O also lies on the perpendicular bisector of side CA. Next, draw the circle with center O and radius $r = OA$, and reason why this is the circumcircle of $\triangle ABC$.

(23) **(Where is the Circumcenter?)** For various triangles ABC, using the procedure above, find where their circumcenters O lie with respect to the triangle, if:
  (a) $\triangle ABC$ is acute (all angles are $< 90°$).
  (b) $\triangle ABC$ is obtuse (one angle is $> 90°$).
  (c) $\triangle ABC$ is right (one angle is $= 90°$).

  Explain why the circumcenter must always lie where you discovered in the above 3 cases.

6. Ultimate Spoilers

(24) **(Circumcenter of a Triangle)** The circumcenter of a triangle lies:
  (a) *inside* the triangle if the triangle is acute.
  (b) *outside* the triangle if the triangle is obtuse.
  (c) *on the hypotenuse* of the triangle if the triangle is right.

(25) **(Cyclic Equilateral Theorem)** Given an equilateral triangle $\triangle ABC$ with its circumcircle k, the distance from a point X on k to the opposite vertex of $\triangle ABC$ equals the sum of the distances from X to the other two vertices. In other words, if X lies on arc $\overset{\frown}{BC}$ that does not contain point A, then $XA = XB + XC$.

(26) **(Ptolemy's Theorem)** Given any cyclic quadrilateral $ABCD$, the sum of the products of opposite sides equals the product of the diagonals, i.e., $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

(27) **(Ptolemy's Thm $\Rightarrow$ Cyclic Equilateral Thm)** Why is Ptolemy's Theorem *stronger* than the Cyclic Equilateral Theorem? Explain how one theorem implies the other.