PLANE GEOMETRY

Reflection, Rotations, and Translations in the Plane, and More

Berkeley Math Circle – Intermediate

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This handout is an extension of the geometry handout from September 2010 on which BMC–beginners worked over a series of 4 geometry sessions. In BMC–intermediate, we shall attempt to cover the same material and move beyond it over 2 geometry sessions. Bring this handout next time!

Try to understand what the problems say and draw pictures for them as best as you can. You are not expected to be able to solve all (or any of) the problems in Section 1 *Overarching Problems and Extensions)* on your own, at least not in the beginning of the geometry sessions at BMC–intermediate. However, you are expected to solve a number of the preparatory exercises after.

All BMC–intermediate students are advised to buy and study carefully Kiselev's Geometry books: first part I *Planimetry* (geometry in 2 dimensions) and then part II *Stereometry* (geometry in 3 dimensions). If you don't yet have copies of these books, ask the BMC–assistant.

Some of the problems in this handout are from *"Mathematical Olympiads"*, part II by Stoyan Budurov and Dimo Serafimov, State Publishing Company *"Narodna Prosveta"*, Sofia, 1985; from the BMC Monthly Contest or from the Bay Area Mathematical Olympiad (BAMO).

Contents

1. Overarching Problems and Extensions

Problem 1. (Three Squares) Three identical squares with bases AM, MH, and HB are put next to each other to form a rectangle $ABCD$. Find the sum of the angles $\angle AMD + \angle AHD + \angle ABD$ and prove that your answer is correct.

Problem 2. (Research Problem) Generalize Problem #1 to 4 squares. Can you find the sum of the resulting four angles? How about the same problem for 5 squares? How about n squares? What happens when we let n go to infinity (i.e., we use an infinite number of squares): will the sum of the angles be a *finite* angle, or will all angles add up to *infinity*?

Problem 3. (Farmer and Cow) During a hot summer day, a farmer and a cow find themselves on the same side of a river. The farmer is 2 km from the river and the cow is 6 km from the river. If each of them would walk straight to the river, they would find themselves 4 km from each other. Unfortunately, the cow has broken its leg and cannot walk. The farmer needs to get to the river, dip his bucket there, and take the water to the cow. To which point on the river should the farmer walk so that his total walk to the river and then to the cow is as short as possible?

Problem 4. (Shortest Broken Line) Two lines p_1 and p_2 intersect. Two points A and B lie in the acute angle formed by the lines. Find a point C on p_1 and a point D on p_2 so that the broken line $ADBCA$ has the smallest possible length. Prove that the points you have found indeed yield this smallest possible length.

Problem 5. (Minimal Perimeter) Given $\triangle ABC$, on the ray opposite to ray $\overrightarrow{CA}$ take a point B_1 so that $|CB_1| = |CB|$. Prove that

(a) Point B_1 is the reflection of B across the angle bisector l of the exterior angle of the triangle at vertex C.

(b) If D is an arbitrary point on l different from C, the perimeter of $\triangle ABD$ is bigger than the perimeter of $\triangle ABC$.

Problem 6. (Find the Perimeter) In $\triangle ABC$, $|AC| = |BC|$ and $|AB| = 10\,\text{cm}$. Through the midpoint D of AC we draw a line perpendicular to AC. This line intersects BC in point E. The perimeter of $\triangle ABC$ is 40 cm. Find the perimeter of $\triangle ABE$.

Problem 7. (Locating Angle Bisector) Two points A and B and a line l are given so that the line intersects segment AB (neither A nor B lies on l). Find point C on l so that the angle bisector of $\angle ACB$ lies on l. Prove that your construction is correct.

Problem 8. (The Bridge in Monthly Contest 1) Two villages A and B lie on opposite sides of a straight river of width d km. There will be a market in village B, which residents of village A wish to attend. To this end, the people of village A need to build a bridge across the river so that the total route walked by the residents of A to the bridge, across the bridge, and onward to B is as short as possible. The bridge, of course, has to be built perpendicular to the river. How can the villagers from A find the exact location of the bridge?

Problem 9. (BAMO '07) In $\triangle ABC$, D and E are two points inside side BC such that $BD = CE$ and $\angle BAD = \angle CAE$. Prove that $\triangle ABC$ is isosceles.

Problem 10. (BAMO '10) Acute $\triangle ABC$ has $\angle BAC < 45°$. Point D lies in the interior of $\triangle ABC$ so that $BD = CD$ and $\angle BDC = 4\angle BAC$. Point E is the reflection of C across line AB, and point F is the reflection of B across line AC. Prove that lines AD and EF are perpendicular.

2. Preparation with Reflections: How to Draw and Basic Properties

(1) What is a *reflection across* line s? Describe in words.

(2) How do we draw reflections of points across s? Draw several examples. Are there any special cases we need to consider? What instruments do we need to draw such reflections precisely? (*Hint:* The answer here includes, among other things, being able to drop a perpendicular from a point to a line.)

(3) **(Parallel axes of reflection)** Lines g_1 and g_2 are parallel. Point M does not lie on them.
   (a) Find the reflection M_1 of M across g_1 and the reflection M_2 of M across g_2.
   (b) If d is the (shortest) distance between g_1 and g_2, prove that $|M_1M_2| = 2d$.

   Consider the case when M is between g_1 and g_2, and the case when M is not between them.

(4) **(For the die-hard explorers)** Suppose in Exercise 3 we keep on reflecting:

$$M \overset{g_1}{\to} M_1 \overset{g_2}{\to} M_2 \overset{g_1}{\to} M_3 \overset{g_2}{\to} M_4 \overset{g_1}{\to} \cdots .$$

   What is the distance $|MM_{2010}|$? How about $|MM_{2011}|$? What additional initial data do you need to know to determine these distances? Explain.

(5) What is a *circle*? Describe carefully and precisely, in words.

(6) Given segment AB, what instruments do you need to find precisely the midpoint M of AB? Describe the construction step by step. (*Hint:* Theorem 2 later on gives away the idea: you need to construct an extra special point C *not* on AB.)

(7) **(Reflections on a circle)** Two lines g_1 and g_2 intersect in point O, and point M does not lie on any of the lines. Let M_1 be the reflection of M across g_1, M_2 the reflection of M_1 across g_2, and M_3 the reflection of M across g_2. Prove that
   (a) points M, M_1, M_2, and M_3 lie on a circle with center O.
   (b) the line determined by O and the midpoint of M_2M_3 is perpendicular to M_2M_3.

Theorem 1. *Segments preserve their lengths under reflection across a line. In other words, if we reflect segment AB across line l to segment A_1B_1, then $|AB| = |A_1B_1|$.*

(8) To prove Theorem 1, consider separately *four* cases, depending on whether segment AB and line l intersect and whether they are perpendicular to each other.

Theorem 2. *In $\triangle ABC$ the segment connecting C to the midpoint M of side AB is perpendicular to AB if and only if $\triangle ABC$ is isosceles with $|AC| = |BC|$.*

(9) Theorem 2 has two directions:
   (a) You assume that $|AC| = |BC|$ and then prove that CM is perpendicular to AB; the latter is written as $CM \perp AB$.
   (b) You assume that $CM \perp AB$ and then prove that $|AC| = |BC|$.

   Prove each direction. Of course, you are allowed to use and obliged to state which criteria for congruent triangles you use.

(10) **(Reflections and angles)** Two lines g_1 and g_2 intersect in point O, and point M does not lie on any of the lines.

(a) Draw the reflection M_1 of M across g_1 and the reflection M_2 of M_1 across g_2.

(b) In case O does not lie on line MM_2, prove that $\angle M_2MO = \angle MM_2O$.

(c) If $|OM| = 9\,\text{cm}$ and $\angle MOM_2 = 120°$, find the distance from O to line MM_2.

Theorem 3. *Angles preserve their measures under reflection across a line, i.e., if points A, B and C are reflected across line l to points A_1, B_1 and C_1, then $\angle ABC = \angle A_1B_1C_1$.*

(11) For a full proof of Theorem 3, you may need to consider a bunch of cases depending on the relative position of $\angle ABC$ with respect to line l. Just concentrate on a generic case, e.g., all three points are on one side of l and none of the sides of $\triangle ABC$ is perpendicular to l.

Theorem 4. *In a* $30°$–$60°$–$90°$ *triangle, the shorter leg is half of the hypothenuse.*
(Note: The shorter leg lies against the $30°$angle. Why?)

(12) One way to prove Theorem 4 is to pick the midpoint of the hypothenuse, show that it is the center of the circle passing through the vertices of the triangle, and see why this introduces a new equilateral triangle in your picture.

(13) **(Reflections and rectangle)** Lines m and n are mutually perpendicular and intersect in point A. Point B does not lie on any of these lines but is in their plane. Point B_1 is the reflection of B across m, point B_2 is the reflection of B_1 across n, and point B_3 is the reflection of B_2 across m.

(a) Draw points B_1, B_2, and B_3.

(b) Prove that points B, B_1, B_2, and B_3 are the vertices of a rectangle.

(c) If point B is at a distance 1.5 cm from line n and segment AB makes an angle of $30°$ with line n, find the length of AB.

Theorem 5. *A quadrilateral is a rectangle if and only if its diagonals intersect each other in a point equidistant from all four vertices. In other words, a quadrilateral $ABCD$ is a rectangle exactly when its diagonals AC and BD intersect in point O such that $|OA| = |OB| = |OC| = |OD|$.*

(14) When proving Theorem 5, note that it requires the proofs of two directions.

(15) What is an *angle bisector* of $\angle ABC$? Describe carefully in words. Assuming we are forbidden to use a protractor, what *other* instruments do we need to draw the angle bisector of angle? Describe the construction step by step.

(16) **(Reflections and angle bisector**) In $\triangle ABC$ the angle bisector of $\angle BAC$ lies on line g_1, while line g_2 is perpendicular to g_1 and passes through point A.

(a) Prove that the angle bisector of the supplementary angle to $\angle BAC$ lies on g_2.

(b) If B_1 and B_2 are the reflections of B across g_1 and g_2, prove that B_1 and B_2 lie on line AC.

(c) Find the length of segment BB_1, given $|AB| = 2.5\,\text{cm}$ and $\angle AB_2B = 30°$.

Theorem 6. *Let α and β be two* supplementary *angles, i.e., α and β share one ray, and their other rays are opposite to each other. Then the* angle bisectors *of α and β are* perpendicular *to each other.*

(17) Draw two *supplementary* angles neither of which is right. Draw their angle bisectors and notice that they seem perpendicular to each other. Prove Theorem 6.

(18) **(Reflecting a whole line)** Given are lines a, b, and s. On line b draw point A_1 so that its reflection across s lies on a. After that locate this reflection of A_1 on line a.

Theorem 7. *The reflection of a line* b *across a line* s *is another line* b_1*. Moreover, either both lines* b *and* b_1 *make the same angle with* s*, or they are both parallel to* s.

(19) Can you think of *two different* solutions to Exercise 18? What does it mean that 3 lines are in *general* position? What special positions for 3 lines can you think of? Investigate what happens in the special relative positions for the lines a, b and s; did your the solution change? What happens if we reflect *all* points of a line b across s? What figure do you get? Do we need to reflect *all* points of b across s in order to find where b goes? Finally, find out and describe all positions of the lines a, b and s in which the exercise has **no** solution. Explain in words what goes wrong and why there is no solution. (*Hint:* One way to look at it is that a certain isosceles triangle will cause trouble. There is another way to describe this troublesome situation, using the word *reflection.*)

Theorem 8 (Triangle Inequality)**.** *Any side of a triangle is shorter than the sum of the other two sides. Moreover, the shortest distance between two points is given by the segment connecting them.*

(20) **(Medians and angle bisector)** In isosceles $\triangle ABC$ ($|AC| = |BC|$) the medians to the congruent sides intersect in point D. Prove that segment DC divides the angle of the triangle at C into two equal parts.

3. Preparation with Circle Geometry

(21) Close your eyes and draw 3 points in the plane. Most likely they won't be *collinear*[1], in other words, most likely they will form a triangle. Why?

Theorem 9 (Circumcircle)**.** *Given $\triangle ABC$, there is exactly one circle that passes through A, B, and C. Consequently, for* three general *points in the plane there will be a unique circle passing through them, called the* circumcircle *of $\triangle ABC$.*

(22) Close your eyes and draw 4 points in the plane. Most likely they won't be *collinear*, and most like they won't be *concyclic*[2] Why? Thus, for 4 points to lie on a circle something special must happen.

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[1] Three points are called *collinear* if they lie on a line.

[2] Four points are called *concyclic* if they lie on a circle.

Theorem 10. *Let A, B, C and D be four points such that C and D are on the same side of line AB. Then $ABCD$ is* cyclic[3] *if and only if $\angle ACB = \angle ADB$. Consequently, for* four general *points in the plane do* not *lie on a circle.*

(23) Draw a trapezoid. Then try to draw a circle passing through the vertices of the trapezoid. Did you succeed? If you pick a *general* ("random") trapezoid, will it be cyclic? Why not?

Theorem 11. *A trapezoid is* cyclic *if and only if the trapezoid is* isosceles. *Consequently, a* general *trapezoid is* not *cyclic.*

(24) Draw a trapezoid and its diagonals. Are the diagonals equal in length? Why or why not?

Theorem 12. *A trapezoid is* isosceles *if and only if its diagonals are* equal *in length.*

4. Preparation with Angles, Calculating Areas, and Trigonometry

(25) Draw $\triangle ABC$. In how many ways can calculate its area? What information about the triangle do you need to know to apply each of your area formulas?

Theorem 13. *We can calculate the area $S_{\triangle ABC}$ of $\triangle ABC$ in at least the following ways:*

(a) $S_{\triangle ABC} = \dfrac{base \cdot height}{2}$, *where the base could be AB and the height – the length of the perpendicular from vertex C to line AB. Note that this formula works even if the foot of this perpendicular lies* outside *side AB.*

(b) $S_{\triangle ABC} = AB \cdot AC \cdot \sin(\angle ABC)$.

(26) Draw an *acute* triangle and then an *obtuse* triangle. In each case, make observations as to which the largest side and largest angle are, which the shortest side and the smallest angle are. Do you notice any correspondence? Is this a coincidence?

Theorem 14. *In a triangle, the larger side lies against the larger angle.*

(27) Draw $\triangle ABC$ and extend side AB beyond point B to point E. Then $\angle EBC$ is called an *exterior* angle for $\triangle ABC$. What is the relation between this exterior angle and the two angle $\angle A$ and $\angle C$ of the triangle? Why does this relation hold?

Theorem 15. *In a triangle, any* exterior *angle equals to the sum of the two* remote interior *angles of the triangle.*

(28) What is the *tangent* of an angle? For which angles is the tangent *undefined* and why?

(29) For which angles α and β does the following famous trigonometric formula *fail*:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}?$$

Why is the formula true for the remainder of the cases?

(30) Can you generalize the above formula to *three* angles? *Four* angles? n angles?

[3] A polygon is *cyclic* if its vertices lie on the same circle, i.e., if there is a circle in which the polygon is *inscribed.*

5. Preparation with Similar and Congruent Triangles

(31) Recall as many criteria for *congruent* and for *similar* triangles as you can. Now state all of them formally and precisely. Your list should include at the very least SSS, SAS, ASA for congruent triangles, and their versions with analogous names SSS, SAS, AA for similar triangles. Did you miss the following two more criteria for congruence:

Theorem 16 (A fourth criteria for triangle congruence)**.** *If two adjacent sides in a triangle are correspondingly equal in length to two adjacent sides in another triangle, and the angle opposite the largest side of the first triangle is equal to the angle opposite the corresponding, also largest, side of the second triangle, then the two triangles are congruent.*

Theorem 17 (A fifth criteria for triangle congruence)**.** *If in a given triangle one of its sides, an angle adjacent to that side, and the sum of the other two sides are correspondingly congruent to the same elements in a second triangle, then the two triangles are congruent.*

(32) State and prove analogous fourth and fifth criteria for similarity of triangles.

6. Mixing Rotations and Translations

(33) What is a *translation* in the plane? What are its basic properties? Describe in words. How about a *rotation* in the plane?

(34) What happens if you apply one rotation in the plane, followed by another rotation in the plane (not necessarily about the same center)? What kind of transformation in the plane is the *composition of two rotations* in the plane?

Theorem 18 (Composition of rotations)**.** *The composition of two rotations in the plane is either a rotation, or a translation. The second case happens iff the sum of the angles of rotation equals* $360°$.

7. Preparation in Logic and Proofs

(35) Physical experiments help us come up with conjectures; but they do not constitute mathematical proofs.

(36) Measuring geometric objects with tools (e.g., ruler, protractor, compass) is a physical experiment, and hence not a mathematical proof. Cutting up and pasting is not a mathematical proof either. Can you think of other useful ways to experiment that cannot be considered proofs?

(37) There may be more than one way to rigorously solve a mathematical problem.

(38) Some problems may look easy, yet their rigorous solution may require non-trivial steps.

(39) Light travels the shortest possible route between objects, even if it must reflect off mirrors.

(40) Extra constructions in a geometry problem are hard to come up with, but a suitable such construction may provide the most beautiful solution to the problem.

(41) Reflections, rotations, translations, and their compositions often inspire extra constructions in geometry problems.