Problem Solving with Polynomials

1 Elementary Properties of Polynomials; Division, Factors, and Remainders

Theorem 1 The **Division Algorithm** for polynomials. If the polynomial $p(x)$ is divided by $d(x)$ then there exist polynomials $q(x), r(x)$ such that

$$p(x) = d(x)q(x) + r(x)$$

and $0 \leq \text{degree}(r(x)) < \text{degree}(d(x))$.

We can find the quotient $q(x)$ and the remainder $r(x)$ by performing ordinary long division with polynomials.

Theorem 2 The **Remainder Theorem** for polynomials. If the polynomial $p(x)$ is divided by $x - a$, then the remainder will be $p(a)$.

Theorem 3 The **Factor Theorem** for polynomials. The polynomial $p(x)$ is divisible by $x - a$ if and only if $p(a) = 0$.

Problems.

1. Find the remainder when $(x+3)^5 + (x+2)^8 + (5x+9)^{1997}$ is divided by $x + 2$.
2. (1974 AHSME #4) Find the remainder when $x^{51} + 51$ is divided by $x + 1$.
3. (1950 AHSME) Find the remainder when $x^{13} + 1$ is divided by $x - 1$.
4. (1999 AHSME #17) The polynomial $P(x)$ has remainder 99 when divided by $x - 19$ and remainder 19 when divided by $x - 99$. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?
5. A polynomial $p(x)$ leaves remainder -2 upon division by $x - 1$ and remainder -4 upon division by $x + 2$. Find the remainder when this polynomial is divided by $x^2 + x - 2$.
6. If $p(x)$ is a cubic polynomial with $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$, find $p(6)$.
7. (1988 AHSME #15) Suppose that a and b are integers such that $x^2 - x - 1$ is a factor of $ax^3 + bx^2 + 1$. What is b?
8. (2003 AMC 12B #9) Suppose that $P(x)$ is a linear polynomial with $P(6) - P(2) = 12$. What is $P(12) - P(2)$?
9. (1977 AHSME #21) For how many values of the coefficient a do the equations
$$0 = x^2 + ax + 1 \text{ and } 0 = x^2 - x - a$$
have a common real solution?
10. Find the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.
11. The polynomial $p(x)$ satisfies $p(-x) = -p(x)$. When $p(x)$ is divided by $x - 3$ the remainder is 6. Find the remainder when $p(x)$ is divided by $x^2 - 9$.
12. (1991 MAΘ) Find all values of m which make $x + 2$ a factor of $x^3 + 3m^2x^2 + mx + 4$.
13. (1982 AHSME) Let $f(x) = ax^7 + bx^3 + cx - 5$, where a, b, c are constants. If $f(-7) = 7$, find $f(7)$.
14. Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Find the remainder when $f(x^5)$ is divided by $f(x)$.

2 Roots and Coefficients

Next, we will consider the **relationship between the zeros of a polynomial and the coefficients of the polynomial**.

Theorem 4 Roots and Coefficients: Suppose that

$$r_1, r_2, \ldots, r_n$$

are the roots of the monic degree-n polynomial

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0.$$

Then, for $k = 1, 2, \ldots n$,

$$a_k = (-1)^{n-k}(\text{sum of all products of } n-k \text{ different zeros}).$$

Problems.

1. (2003 AMC 10A #18) What is the sum of the reciprocals of the roots of the equation
$$\frac{2003}{2004}x + 1 + \frac{1}{x} = 0?$$

2. (2005 AMC 10B #16) The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, p is zero. What is the value of n/p?

3. (2006 AMC 10B #14) Let a and b be the roots of the equation $x^2 - mx + 2 - 0$. Suppose that $a + (1/b)$ and $b + (1/a)$ are the roots of the equation $x^2 - px + q = 0$. What is q?

4. (2001 AMC 12 #19) The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. The y-intercept of the graph of $y = P(x)$ is 2. What is b?

5. (1963 AHSME #14) Consider the equations $x^2 + kx + 6 = 0$ and $x^2 - kx + 6 = 0$. If, when the roots of the equations are suitably listed, each root of the second equation is 5 more than the corresponding root of the first equation, find k.

6. (2000 AMC 10 #24) Suppose that $P(x/3) = x^2 + x + 1$. What is the sum of all values of x for which $P(3x) = 7$?

7. (1983 AIME) What is the product of the real roots of the equation
$$x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}?$$

8. (1984 USAMO) The product of two of the four zeros of the quartic equation
$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$
is -32. Find k.

9. If three roots of $x^4 + Ax^2 + Bx + C = 0$ are $-1, 2, 3$, then what is the value of $2C - AB$?

10. Find the largest solution of
$$x^3 - 27x^2 + 242x - 270 = 0$$
given that one root equals the average of the other 2 roots.

11. (1991 MAΘ) For nonzero constants c and d, the equation $4x^3 - 12x^2 + cx + d = 0$ has 2 real roots which add to 0. Find d/c.

12. (1977 USAMO) If a and b are two roots of $x^4 + x^3 - 1 = 0$, show that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

3 Transformations of Polynomials

In this section, we will study the following questions:

- Given a polynomial with roots $r_1, \ldots, r_n$, how do we find a polynomial with roots $\frac{1}{r_1}, \ldots, \frac{1}{r_n}$?
- Given a polynomial with roots $r_1, \ldots, r_n$, how do we find a polynomial with roots $kr_1, \ldots, kr_n$, where k is a given scalar?
- Given a polynomial with roots $r_1, \ldots, r_n$, how do we find a polynomial with roots $r_1 + k, \ldots, r_n + k$, where k is a given scalar?

Problems.

1. Find the polynomial of minimal degree with integer coefficients whose roots are the reciprocals of the roots of $f(x) = x^2 - 5x + 6 = 0$.
2. Find the polynomial of minimal degree with integer coefficients whose roots are the reciprocals of the roots of $f(x) = x^4 - 3x^2 + x - 9$.
3. Find a polynomial of minimal degree with integer coefficients whose roots are twice those of $f(x) = x^2 - 5x + 6$.
4. Find a polynomial of minimal degree with integer coefficients whose roots are twice those of $f(x) = x^4 - 3x^2 + x - 9$.
5. Find a polynomial of minimal degree with integer coefficients whose roots are half the reciprocals of the roots of $5x^4 + 12x^3 + 8x^2 - 6x - 1$.
6. Find a polynomial of minimal degree with integer coefficients whose roots are 3 greater than those of $f(x) = x^4 - 3x^3 - 3x^2 + 4x - 6$.
7. The roots of $f(x) = 3x^3 - 14x^2 + x + 62 = 0$ are a, b, c. Find the value of
$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}.$$
8. (1981 AHSME) If a, b, c, d are the solutions of the equation $x^4 - mx - 3 = 0$, find the polynomial with leading coefficient 3 whose roots are
$$\frac{a+b+c}{d^2}, \frac{a+b+d}{c^2}, \frac{a+c+d}{b^2}, \frac{b+c+d}{a^2}.$$
9. (1991 MAΘ) Let r, s, t be the roots of $x^3 - 6x^2 + 5x - 7 = 0$. Find
$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2}.$$
10. If $p(x)$ is a polynomial of degree n such that $p(k) = 1/k, k = 1, 2, \ldots, n+1$, find $p(n+2)$.