Introduction to Sound Analysis

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Berkeley Math Circle, Intermediate

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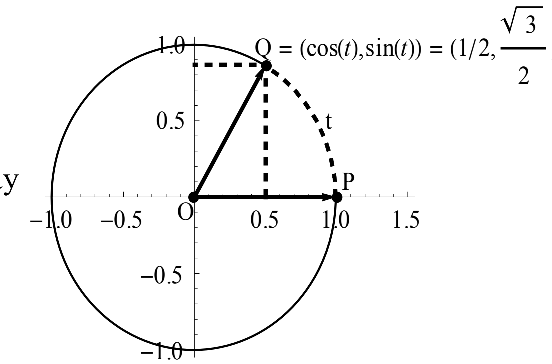
Summary: We talked about some of the math our brain does when we listen to sounds. Because we are able to hear many characteristics of functions, linking our senses of sound and math helps us appreciate and understand both. You can see this works because many mathematicians are also avid musicians!
Along the way, we discussed trigonometric functions like sine(x) and how these can be used as good building blocks to construct other periodic functions. This handout only highlights topics – it is not meant to be detailed enough to learn the subject from scratch.

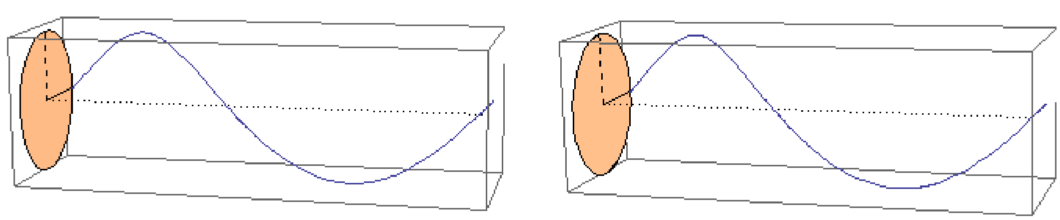
The last pages contains problems. Solutions can be turned in to BMC by the session on 2010.10.19. If any are (partial solutions ok), an **award of $50 000 000 000 000** will be given to one of the best (determined by lottery.) Answers will be available on 2010.10.26. [Caveat: to receive the award, you have to attend on 2010.10.26, and you have to be able to explain your work. Barring sufficient hyperinflation in the next two weeks, the award might be in a non-US currency.]

The basic functions:

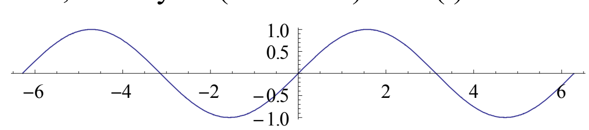


Recall that sin(t) (called "sine") and cos(t) (called "cosine") are the y- and x- coordinates of a point that has moved counterclockwise around a unit circle by t radians from the starting point of (1,0). In math, radians are a more useful way to measure angles than degrees, because they measure arc length around a unit circle (a circle with radius 1.) So in the example at right, the angle POQ is t = $\pi/3$ = 60 degrees. We see the dashed arc traced out by the angle t, and at the point Q(t), we can read off the coordinates to see that cos(t) = ½ and sin(t) = $\sqrt{(3)}/2$.

As t goes around the unit circle by increasing from 0 to 2π , Q(t) traces out a spiral or helix. You can see this in the 3d-stereogram below if you manage to focus your left eye on the left graph while focusing your right eye on the right one. [Hint: try getting your eyes to merge the two shaded disks.] In each graph, the positive t-axis (dotted) goes from left to right, the shaded unit circle sits in the x-y plane at left (where t=0), the positive x-axis (solid line) points into the page, and the positive y-axis (dashed) points up.



Since this is hard to visualize, let's focus only on the dimensions t and y = sin(t). This time, you can see that y "repeats itself" after t increases by the **period** of 2π , namely sin(t + 2π) = sin(t).

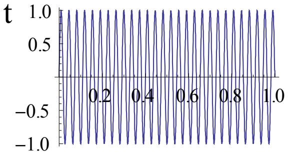


Here you can also see that sin(t) is an **odd function**, namely sin(-t) = - sin(t). Other odd functions include y(t) = t^n for n odd. By contrast, even functions satisfy y(t) = y(-t), like even powers of t.

Question: If t is measured in seconds, what does sin(t) sound like?

Answer (to us anyway): we hear nothing, because the frequency = 1/period = 1/(2π seconds). This is less than 1/6 Hz, where "Hz" stands for Hertz, meaning periods (or cycles) per second.

To create a function we can hear, before applying the function sine, we multiply t by 2π times the frequency we want. For example y(t) = sin(2π 440 t) is the "a" above middle c, written "A4." Since it is hard to graph so many wiggles, we instead graph one second of the lowest frequency on a piano (27.5 Hz, written "A0".) This is 4 octaves lower, and so has a frequency 2^4 times lower: y(t) = sin(2π 27.5 t).



Note that we have called this the lowest "frequency" rather than the lowest "note". Sounds we hear as single notes usually include "overtone" frequencies that are multiples of the base frequency. So an A0 played on the piano will also include the following (and others):

Multiple of base frequency	1	2	3	4	5	6	7	8	...
frequency	27.5	55	82.5	110	137.5	165	192.5	220	
pitch	A0	A1	roughly E2	A2	roughly C#3	roughly E3	roughly G3	A3	

Notice that only octaves are exact integer ratios. All other intervals are powers of half note intervals, which are $2^{(1/12)} \approx 1.06$, because they come from dividing an octave (ratio 2) into 12 intervals. For example, a perfect fourth interval like from E2 to A2 is 5 half notes, so the ratio of frequencies is $2^{(5/12)} \approx 1.06^5 \approx 1.34$. This is close enough to the ratio 4/3 ≈ 1.33 that our hearing perceives it as a harmony.

When two frequencies played together are not near integer multiples of each other, we perceive them as different notes. For example, "touch-tone" telephone sounds are made by combining one note for each row with one note for each column, so a single telephone button plays a combination of two notes. (To hear a single one, push simultaneously two buttons in a single row or in a single column.)

Understanding transformations applied before and after applying a function:

Given constants a, b, c, and d, the function y(t) = c*sin(a*x+b)+d may seem complicated to graph. But a closer look at the formula shows you can separate what is done before applying sin from what happens afterwards:

First, t → a*t + b → sin(a*t+b) [let's call this "u".] Then u → c*u + d.

Anything you do before applying sine can be thought of as affecting the (horizontal) t-axis. For example, we have seen how sin(2 t) has half the period as sin(t.)

Similarly, any operation performed after applying sine can be thought of in terms of the (vertical) y-axis. For example, y(t) = 2*sin(t) would have twice the amplitude of sin(t). We would perceive the larger vertical changes as sounding louder, but at the same frequency.

We listened to many other combinations of sine waves–for more see the problems on the next two pages

Superposition: two sounds can be "played together" just by adding their functions. You can get very complicated patterns just by adding two sine functions of different frequencies. The amazing thing is that you can get ANY ("well behaved") function of period 2π by just adding a constant and multiples of functions like y(t) = sin(n*t) and z(t) = sin(n*t + $\pi/2$) for n a positive integer. The catch is you may have to add infinitely many terms for a perfect fit!

FM synthesis: you change a base ("carrier") frequency by varying the frequency INPUT to the outer sine function (the one in capitals): SIN(2π 880 t + 10 sin(2π t)), This sounds like a vibrato, or like a police siren if you increase the 10 to say 100.)

AM synthesis: here you change the OUTPUT of the base frequency by multiplying by a function with lower frequency: y(t) = [2 + sin(2π t)]*sin(2π * 1000 t). This gets quieter and louder.

White noise mixes random frequencies. If you multiply it by a sine wave, it sounds like breathing.

Problems

1) In the unit circle of the first diagram, apply the Pythagorean theorem to a triangle with hypotenuse OQ to derive a relation between sin(t) and cos(t). Use this to find a formula of cos(t) in terms of sin(t).

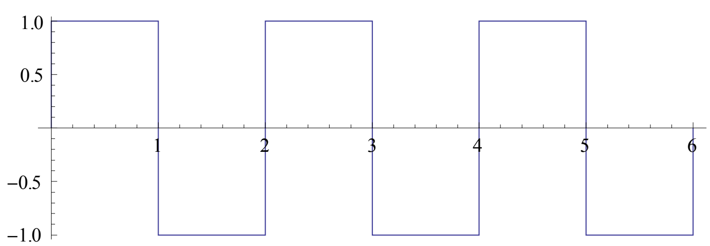
3) For most angles t, one cannot find algebraic formulas for sin(t) and cost(t). But show you can do this for t = $\pi/3$ radians = 60 degrees by using the first diagram. [Hint: show OPQ is equilateral. Then use the fact that the line from (cos(t), 0) to Q is an altitude for this triangle.]

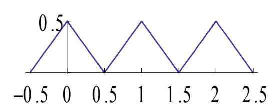
2) Graph y(t) = cos(t). Is it even or odd? If so, which? If not, why not?

3) You already found one formula for cos(t) in terms of sin(t) in problem 1. Try to find solutions of the form cos(t) = c*sin(a*t + b) + d for constants a, b, c, d where a, b, and c are not the same for each solution. What does d have to be?

4) There are many other periodic functions we could try using instead of sine waves, like:

squareWave(t) and sawtooth(t)





For each of these, find a formula for a function closest to what is graphed, indicate if it is odd or even, and find a function of the form c*sin(a*t) + d most similar to the function

[Hint: the similar sine should have the same period. An even better match would also try to match amplitude, whether it is odd or even, and “average value” (always positive? Symmetric about x-axis?)]

[Hint 2: the formula for sawtooth(t) is tricky. Try ingredients like abs(t), t – floor(t), etc.]

5) For each of the graphs below, match each formula that describes it. If a formula or graph has no matching counterpart, add one and identify it.

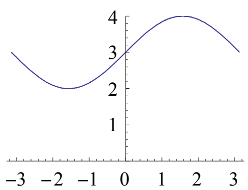
(i) sin(t) + 3 (ii) sin(t) + sin(3 t) (iii) sin(t) – sin(3 t)

(iv) $t - t^3/3! + t^5/5!$ [Hint: these are the first three terms in an infinite polynomial that is exactly equal to sin(t) for all t. But if you take only these terms, the value is only close to sin(t) for t near 0.]

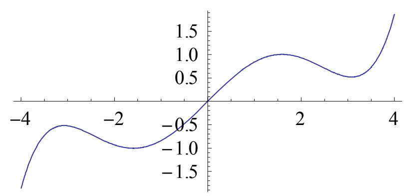
(v) 2*sawtooth(t)*sin(2π *6 t) (vi) sin(t - π) (vii) – sin(t)

(viii) (for a missing formula) ________________

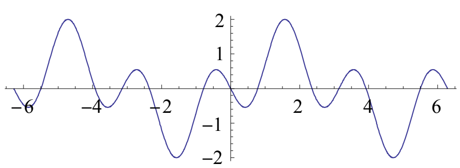
a)



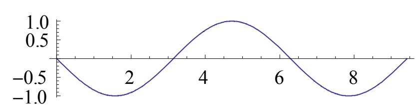
b)



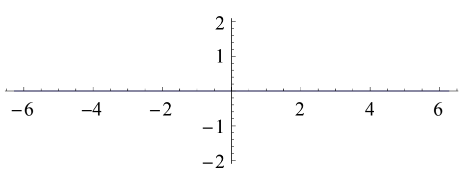
c)



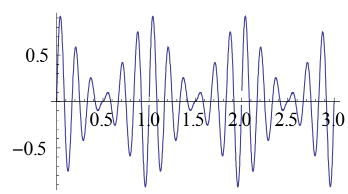
d)



e) (for adding a missing graph)



f)



6a) Problem 5.iv above shows how you can begin to assemble a sine curve (which has infinitely many "wiggles") out of polynomials (each terms of which contributes only one "wiggle.") Do you think you could add multiples of infinitely many functions sin(2π *n*t) (for integer n) to make a polynomial like t^3 ? Why or why not?

6b) It turns out you <u>can</u> add multiples of functions sin(2π *n*t) (for integer n) to make a function f(t) such that f(t) = t^3 for t strictly between -1/2 and +1/2. Given this information, can you figure out what f(3) would be?