GEOMETRY PROBLEMS
via Complex Numbers and Geometric Transformations

Berkeley Math Circle – Advanced

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1. Preparatory Plane Geometry Problems given to BMC-Intermediate

Problem 1. (Three Squares) Three identical squares with bases AM, MH, and HB are put next to each other to form a rectangle $ABCD$. Find the sum of the angles $\angle AMD + \angle AHD + \angle ABD$ and prove that your answer is correct.

Problem 2. (Research Problem) Generalize Problem #1 to 4 squares. Can you find the sum of the resulting four angles? How about the same problem for 5 squares? How about n squares? What happens when we let n go to infinity (i.e., we use an infinite number of squares): will the sum of the angles be a *finite* angle, or will all angles add up to *infinity*?

Problem 3. (Farmer and Cow) During a hot summer day, a farmer and a cow find themselves on the same side of a river. The farmer is 2 km from the river and the cow is 6 km from the river. If each of them would walk straight to the river, they would find themselves 4 km from each other. Unfortunately, the cow has broken its leg and cannot walk. The farmer needs to get to the river, dip his bucket there, and take the water to the cow. To which point on the river should the farmer walk so that his total walk to the river and then to the cow is as short as possible?

Problem 4. (Shortest Broken Line) Two lines p_1 and p_2 intersect. Two points A and B lie in the acute angle formed by the lines. Find a point C on p_1 and a point D on p_2 so that the broken line $ADBCA$ has the smallest possible length. Prove that the points you have found indeed yield this smallest possible length.

Problem 5. (Minimal Perimeter) Given $\triangle ABC$, on the ray opposite to ray $\overrightarrow{CA}$ take a point B_1 so that $|CB_1| = |CB|$. Prove that

(a) Point B_1 is the reflection of B across the angle bisector l of the exterior angle of the triangle at vertex C.

(b) If D is an arbitrary point on l different from C, the perimeter of $\triangle ABD$ is bigger than the perimeter of $\triangle ABC$.

Problem 6. (Find the Perimeter) In $\triangle ABC$, $|AC| = |BC|$ and $|AB| = 10\,\text{cm}$. Through the midpoint D of AC we draw a line perpendicular to AC. This line intersects BC in point E. The perimeter of $\triangle ABC$ is $40\,\text{cm}$. Find the perimeter of $\triangle ABE$.

Problem 7. (Locating Angle Bisector) Two points A and B and a line l are given so that the line intersects segment AB (neither A nor B lies on l). Find point C on l so that the angle bisector of $\angle ACB$ lies on l. Prove that your construction is correct.

Problem 8. (The Bridge in Monthly Contest 1) Two villages A and B lie on opposite sides of a straight river of width d km. There will be a market in village B, which residents of village A wish to attend. To this end, the people of village A need to build a bridge across the river so that the total route walked by the residents of A to the bridge, across the bridge, and onward to B is as short as possible. The bridge, of course, has to be built perpendicular to the river. How can the villagers from A find the exact location of the bridge?

2. Geometry Problems Strictly for BMC-Advanced

Problem 9. (BAMO '07) In $\triangle ABC$, D and E are two points inside side BC such that $BD = CE$ and $\angle BAD = \angle CAE$. Prove that $\triangle ABC$ is isosceles.

Problem 10. (BAMO '10) Acute $\triangle ABC$ has $\angle BAC < 45°$. Point D lies in the interior of $\triangle ABC$ so that $BD = CD$ and $\angle BDC = 4\angle BAC$. Point E is the reflection of C across line AB, and point F is the reflection of B across line AC. Prove that lines AD and EF are perpendicular.

Problem 11. (MOSP '99 IMO test) Let H, O, and R be the orthocenter, circumcenter, and circumradius of $\triangle ABC$. Let A_1, B_1, and C_1 be the reflections of A, B, and C across lines BC, CA, and AB. Prove that A_1, B_1, and C_1 are collinear iff $|OH| = 2R$.

Problem 12. (Bulgaria '97) Let the convex quadrilateral $ABCD$ be inscribed in a circle. Let $F = AC \cap BD$ and $E = AD \cap BC$. If M and N are the midpoints of AB and CD, prove that

$$\frac{MN}{EF} = \frac{1}{2}\left|\frac{AB}{CD} - \frac{CD}{AB}\right|.$$