Berkeley Math Circle April 26, 2011

Instructor: Brandy Wiegers, brandy@msri.org

1 Mathemagical Black Holes

The physical universe contains strange objects called black holes, whose gravitational field is so powerful not even light can escape. The mathematical universe also contains black holes, seemingly innocuous numbers from which no other number can escape . . .

The Sisyphus Number

To find a Sisyphus number:

- Choose any positive whole number and write it down.
- Count the number of even digits, the number of odd digits, and the total number of digits.
- Record these digits (including zero if it is one of your values) as a new number.
- Repeat this process.
- If a number given by this process repeats, it is a Sisyphus number. For example, choosing 44,987, we count 3 even digits, 2 odd digits, and 5 total digits. Our resulting number would be 325. We then repeat the process until we find a stable result, the Sisyphus number.

Starting Number	# of EVEN digits	# ODD digits	Total # of digits	Resulting Number

Once youve found a Sisyphus number, compare with your neighbors. Notice anything?
Homework: Show why only one Sisyphus number results from this process.

Words to Numbers

- Choose any positive whole number and write it down.
- Write out its numeral in English, such as TWENTY-THREE for 23.
- Count the number of characters in the spelling (including spaces and hyphens), for example, TWENTY-THREE has 12 characters.
- Repeat this process using the number of characters. What mathematical black hole draws every number in using this process? Is there such a number in Spanish? Or any other language you know?

The Narcissis Number

- Choose a positive multiple of three and write it down. (Recall that the digits of any multiple of three sum to a multiple of threethis makes it easy to produce large multiples of three.)
- Take the cube of each digit of your number. (A calculator may be useful here.)
- Add up the cubes to form a new number.
- Repeat the process.

Continued repetition will take you to another black hole, the Narcissus number. What is it? Can you prove that any multiple of three will be drawn into this black hole?

Two more black holes to investigate . . .

Kaprekar's Constant

- Choose any four-digit number other than an integer multiple of 1111. That is, do not select 1111, 2222, 3333, . . ., or 9999.
- Rearrange the digits of your number from largest to smallest and record this new number.
- Rearrange the digits of your number from smallest to largest and sub- tract this number from the number you wrote down in the previous step.
- Repeat the process to find Kaprekar's constant.

A Divisive Number

- Write down any positive whole number.
- Write down all of the numbers divisors including 1 and itself.
- Sum the digits of the divisors.
- Repeat the process to find this divisive number.

[Adapted by Ralf Youtz from Michael W. Ecker's Number Play, Calculators, and Card Tricks: Mathemagical Black Holes in The Mathemagician and Pied Puzzler: A Collection in Tribute to Martin Gardener.]

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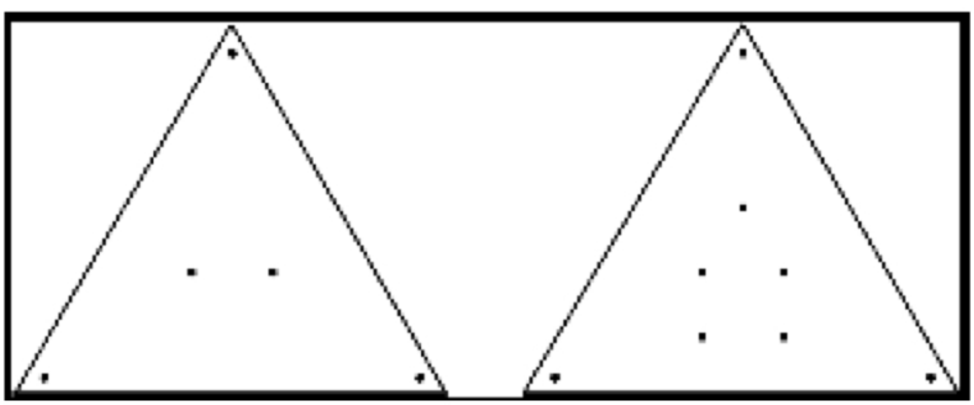
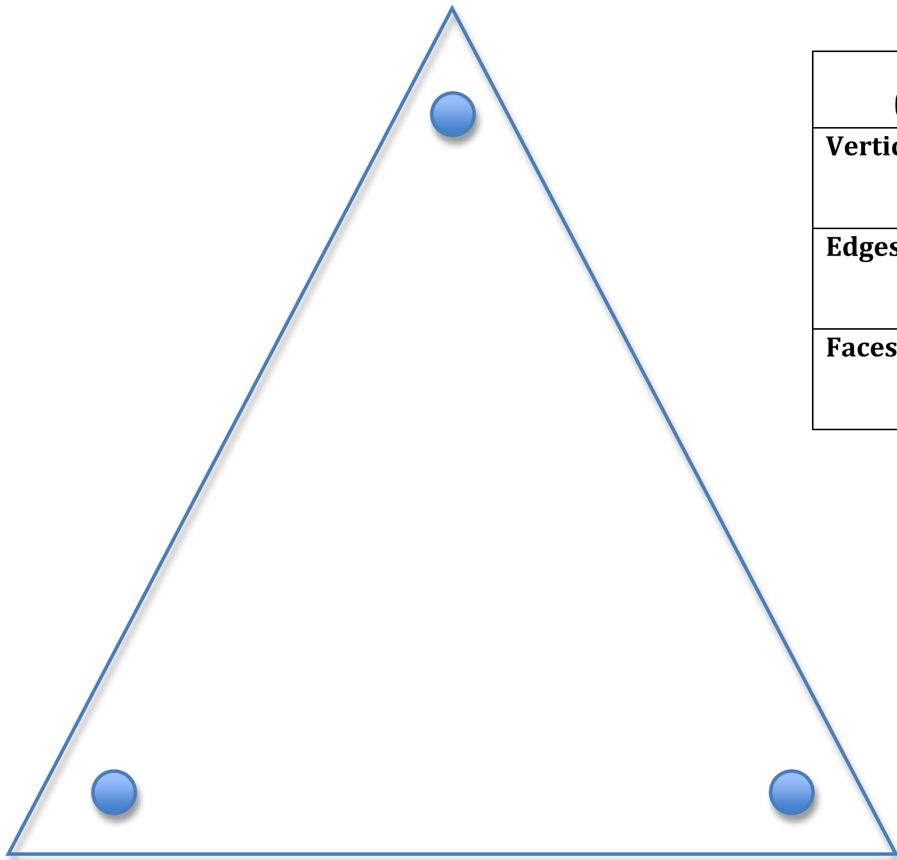
2 The Game of Criss Cross

In this fascinating game of skill and strategy, players alternate turns until one of the players is unable to make a move. To begin, create a game board by drawing up to seven more dots inside the triangle. Then decide who will play first. On each turn, draw a segment between any pair of dots which does not intersect any of the existing segments. The winner is the last person able to make a legal move.

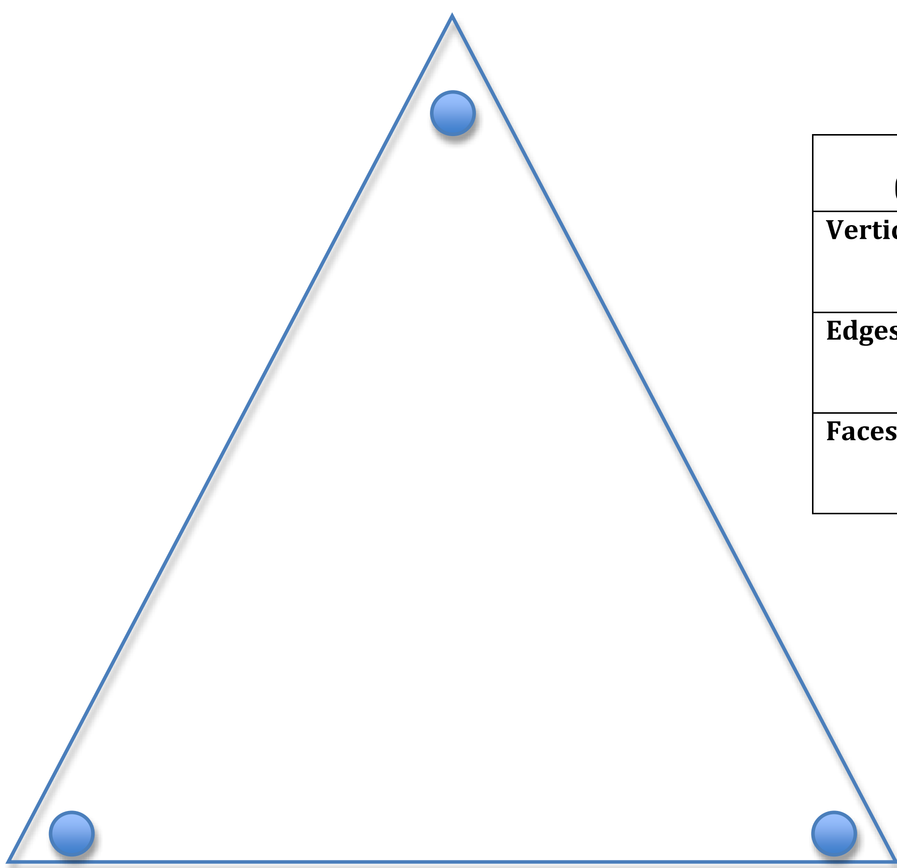


Two sample Criss-Cross boards

The game board is created by drawing three points at the vertices of a large equilateral triangle, along with two to seven additional points anywhere in its interior. Two sample boards are shown above. Players alternate turns drawing a single straight line segment joining any two points, as long as the segment does not pass through any other points or segments already appearing on the game board. The winner is the last player able to make a legal move.



Who Won? (Circle one)	First Player Second Player
Vertices (V)	
Edges (E)	
Faces (F)	



Who Won? (Circle one)	First Player Second Player
Vertices (V)	
Edges (E)	
Faces (F)	

Questions to Consider

1. How many different moves can the first player make on the game board pictured at left above?
2. Will the first or second player win on the left-hand game board? Explain why any game on this board always lasts for the same number of moves.
3. Play three games of Criss-Cross using the right-hand game board. Make a conjecture regarding the outcome of any game played on this board.
4. By trying games on several different boards, come up with a method of predicting the winner of any game of Criss-Cross based on the board configuration.
5. Use reasoning similar to the previous problem to prove that your method of predicting the winner on any game board is valid.

Further Questions to Consider

1. Based on a particular game board configuration, how can you predict which player will win that game before the first move is made?
2. Now examine the three quantities listed at the bottom of each game board. What relationships seem to always hold among these three numbers? Hint a. Which quantity seems to always be the largest?
   Hint b. How much larger is it than the others?
   Hint c. Make a plot of F, the number of faces (regions) versus E, the number of edges? (segments). What do you notice? Can you find an equation relating these? two quantities?
3. Why is it always the case that every region on a completed game board will be triangular?
4. Now demonstrate that it is always the case that 3F=2E-4 on a completed game board. (Hint: imagine cutting out the regions and counting the resulting edges of all the pieces. Then paste the edges back together again to reassemble the board.)
5. Use the two formulas you now have (namely V-E+F=1 and 3F=2E-4) to obtain a relationship between V and E.
6. How many edges will there be on a completed game board with 100 points all together? (Including the four points at the corners of the square.) Which player will win this game, and why?
7. Prove that your conjecture from the first question above is true.
8. In a certain small country there are villages, expressways, and fields. Expressways only lead from one village to another and do not cross one another, and it is possible to travel from any village to any other village along the expressways. Each field is completely enclosed by expressways and villages. If there are 100 villages and 141 expressways, then how many fields are there?
9. Suppose that we change the outer boundary of a Criss-Cross board so that it consists of four points at the corners of a square. Play games in which there are either one, two, or three additional points in the interior of this square. Based on the outcomes, make a conjecture regarding how to predict whether the first or second player will win on a square board.

3 Bulgarian Solitaire

Bulgarian solitaire, also known as deterministic Bulgarian solitaire, was invented by the famous American recreational mathematician Martin Gardner in 1983.

- Start with 15 coins: ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛
- Move the coins into any amount of stacks of any size so that you have n stacks with 1-15 coins in each stack. Example: 1 stack of 15 coins or 15 stacks of 1 coin each or 5 stacks of 3 coins, etc..
  Question: How many possible stack/ coin combinations could you start with?
- Now, a game move consists of taking one coin from each stack that exists at the beginning of the move and creating a new stack with these coins.

  Example:

  Start of the move:
  ⊛ ⊛ ⊛ ⊛ ⊛
  ⊛ ⊛ ⊛ ⊛ ⊛ ⊛ ⊛
  ⊛ ⊛
  ⊛

  End of the move:
  ⊛ ⊛ ⊛ ⊛
  ⊛ ⊛ ⊛ ⊛ ⊛ ⊛
  ⊛

  ⊛⊛ ⊛ ⊛

  Notes about moves:

  - If you start a move with 1 coin in a stack then that stack disappears as the coin joins the new stack.
  - As you do a move and create a new stack, that stack should be included and used during the next move.
- Continue to do this until you have a set of stacks which has already occurred.

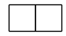
Questions to Consider

1. How many steps does it take for the game to end?
2. Does this depend on your starting stacks?
3. How do you know the game will eventually end?
4. What happens if you start with a different number of coins?
5. Will you always get a pattern that will repeat at each step?If not, how many steps can it take for your pattern to repeat?

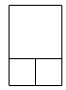
References Akin, E. & Davis, M. (1984) 'Bulgarian Solitaire', in American Mathematical Monthly; (2); 92. Pages 237-250. Bentz, H.-J. (1987) 'Proof of the Bulgarian Solitaire Conjectures', in Ars Combinatoria; 23. Pages 151-170. Gardner, M. (1983) 'Mathematical Games. (a.k.a Bulgarian Solitaire and Other Seemingly Endless Tasks)', in Scientic American; 249. Pages 8-13 or 12-21. Gwihen, E. (1991) 'Tableaux de Young et Solitaire Bulgare', in Journal of Combinatorial Theory; (2); 58. Pages 181-197. Hobby, J. D. & Knuth, D. (1983) 'Problem 1: Bulgarian Solitaire', in A Programming and Problem-Solving Seminar, Stanford: Department of Computer Science, Stanford University; (December). Pages 6-13. Igusa, K. (1985) 'Proof of the Bulgarian Solitaire Conjecture', in Mathematical Magazine; (5); 58. Pages 259-271. Nicholson, A. (1993) 'Bulgarian Solitaire', in Mathematics Teacher; 86. Pages 84-86.

A Special Kind of Rectangle

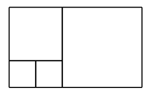
Start with two adjoined 1x1 boxes



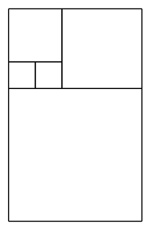
On top of both of these draw a square of size 2 (=1+1).



We can now draw a new square - touching both a unit square and the latest square of side 2 - so having sides 3 units long;



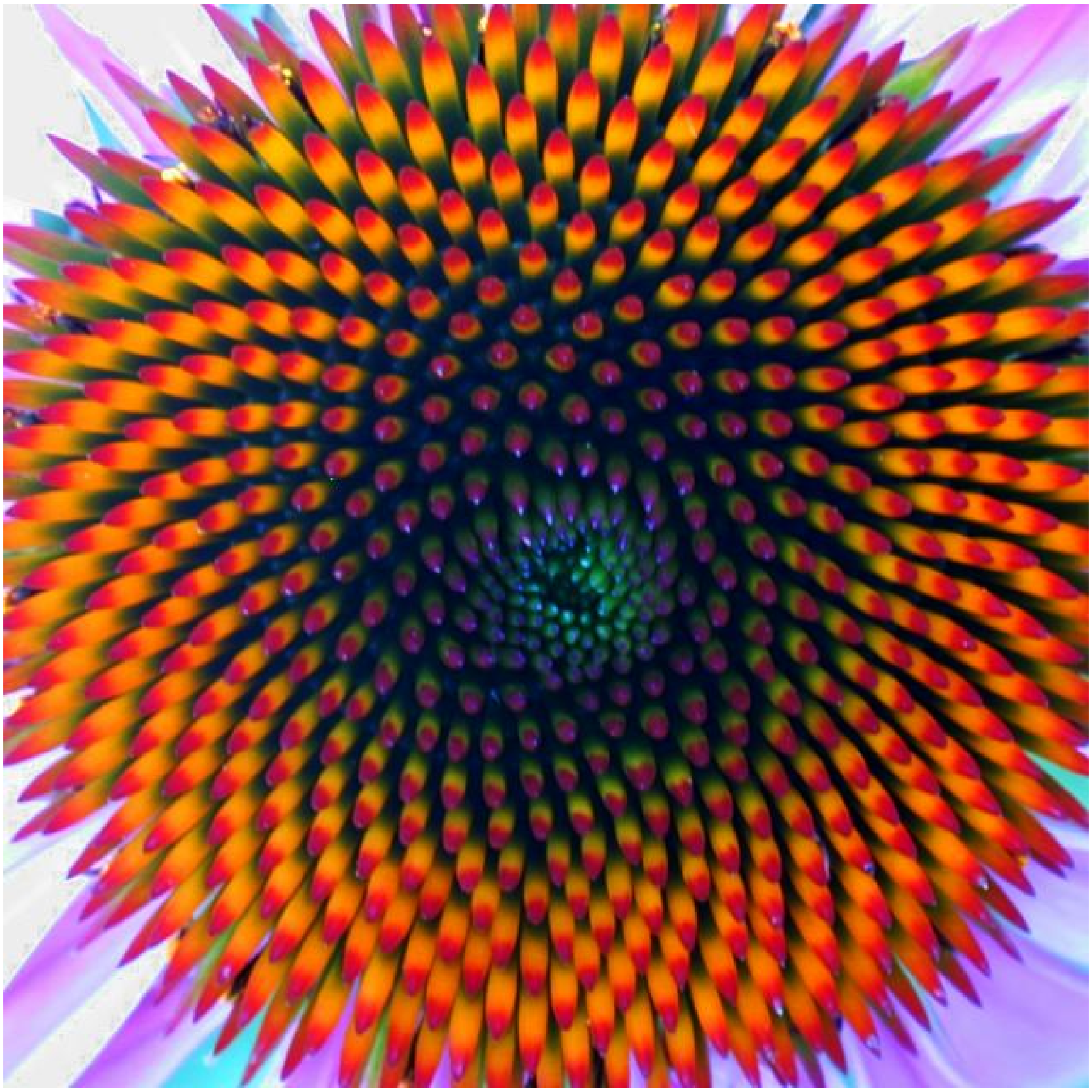
and then another touching both the 2-square and the 3-square (which has sides of 5 units).



Questions to Consider?

- What size do you think the next several squares will be? What will the resulting rectangle look like?

- Do you see a pattern
- Have you seen this pattern before?
- How do you think this related to mathematics or biology?



Questions to Consider?

- Do you see a pattern
- Have you seen this pattern before?
- How do you think this related to mathematics or biology?