Berkeley Math Circle
Monthly Contest 8
Due May 3, 2011

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 8, Problem 3
by Bart Simpson
in grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Let x and y be positive integers. Can $x^2 + 2y$ and $y^2 + 2x$ both be squares?

2. On a 5×5 chessboard, a king moves according to the following rules:

   - It can move one square at a time, horizontally, vertically, or diagonally. (These are the usual moves of the king in chess.)
   - It can move in each of the eight allowable directions at most three times in its entire route.

   The king can start at any square. Determine

   (a) whether the king can visit every square;

   (b) whether the king can visit every square except the center.

3. Define the *digitlength* of a positive integer to be the total number of letters used in spelling its digits. For example, since "two zero one one" has a total of 13 letters, the digitlength of 2011 is 13. We begin at any positive integer and repeatedly take the digitlength. Show that after some number of steps, we must arrive at the number 4.

4. Suppose 2011 light bulbs are arranged in a row. Each bulb has a button under it. Pressing the button will change the state of the bulb above it (on to off or vice versa) and will also change the two neighboring bulbs, or the single neighboring bulb in the case of one of the two end buttons. Is it always possible, regardless of the initial state of the bulbs, to turn them all off by pressing some buttons?

5. Let ABC be a triangle with $\angle ACB = 90°$. The inscribed circle of $\triangle ABC$ touches sides AC and BC at D and E, respectively. On the circumscribed circle of $\triangle ABC$, the midpoints of minor arcs AC and BC are respectively P and Q. Prove that D, E, P, and Q are all collinear.

6. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that $f(0) = 2$ and for all integers x,

   $$f(x + 1) + f(x - 1) = f(x)f(1).$$

   Prove that for all integers x and y,

   $$f(x + y) + f(x - y) = f(x)f(y).$$

7. Let n be a positive integer which is divisible by 5 and which can be written as the sum of two (not necessarily distinct) squares. Prove that n can be written as the sum of two squares one of which is greater than or equal to four times the other.

   Remark. A *square* is the square of any integer including zero.