# Berkeley Math Circle
# Monthly Contest 7 – Solutions

1. Let $x$ and $y$ be integers such that $\frac{3x+4y}{5}$ is an integer. Prove that $\frac{4x-3y}{5}$ is an integer.

   *Solution.* The basic strategy is to combine the facts that $x$, $y$, and $(3x+4y)/5$ are all integers. Here is one solution:

$$2(x) + 1(y) - 2\left(\frac{3x+4y}{5}\right) = \frac{5(2x+y) - 2(3x+4y)}{5} = \frac{10x+5y-6x-8y}{5} = \frac{4x-3y}{5}.$$

   Since the left side is clearly an integer, so is the right side.

2. Six rooks are placed on a $6 \times 6$ chessboard, at the locations marked $+$, so that each rook "attacks" the five squares in the same row and the five squares in the same column. Determine if it is possible to label each empty square with a digit (0 through 9) so that for each rook, the ten squares which it attacks are all labeled with different digits.

   *Solution.* The answer is yes. One of many solutions is shown.

| | | | | | |
|---|---|---|---|---|---|
| + | 5 | 6 | 7 | 8 | 9 |
| 0 | + | 7 | 8 | 9 | 6 |
| 1 | 2 | + | 9 | 5 | 8 |
| 2 | 3 | 4 | + | 6 | 5 |
| 3 | 4 | 0 | 1 | + | 7 |
| 4 | 1 | 3 | 0 | 2 | + |

3. Let $f(n)$ be the number of digits of a positive integer $n$ (in base 10). Prove that

$$f(2^n) + f(5^n) = n + 1.$$

   *Solution.* Let $f(2^n) = x$. Then since the smallest number with $x$ digits is $10^{x-1}$ and the largest is $10^x - 1$, we have

$$10^{x-1} \leq 2^n < 10^x.$$

   However, a power of 2 (other than 1) cannot also be a power of 10, so the inequality is strict:

$$10^{x-1} < 2^n < 10^x. \quad (1)$$

   Similarly, if $f(5^n) = y$, then

$$10^{y-1} < 5^n < 10^y. \quad (2)$$

   Multiplying (1) by (2),

$$10^{x-1} \cdot 10^{y-1} < 2^n \cdot 5^n < 10^x \cdot 10^y$$
$$10^{x+y-2} < 10^n < 10^{x+y}$$
$$x + y - 2 < n < x + y.$$

   Since the only integer between $x+y-2$ and $x+y$ is $x+y-1$,

$$n = x + y - 1$$

   so $x + y = n + 1$ as desired.

4. Let $ABC$ be a triangle with incenter $I$. A line through $I$ parallel to $BC$ intersects sides $AB$ and $AC$ at $D$ and $E$ respectively. Prove that the perimeter of $\triangle ADE$ is equal to $AB + AC$.

   *Solution.* Because $DE \parallel BC$, $\angle BID = \angle IBC$ which is the same as $\angle DBI$ since $I$ is on the bisector of $\angle ABC$. Thus $\triangle BDI$ is isosceles, implying $BD = DI$. Similarly $CE = EI$. Thus the perimeter of $\triangle ADE$ is

$$AD + AE + DE = AD + AE + DI + EI = AD + AE + BD + CE = AB + AC.$$

5. Two dice are loaded so that the numbers 1 through 6 come up with various (possibly different) probabilities on each die. Is it possible that, when both dice are rolled, each of the possible totals 2 through 12 has an equal probability of occurring?

*Solution.* The answer is no. Suppose that each of the totals from 2 to 12 has an equal probability, which must be $1/11$ since the sum of all probabilities is 1. Let $a$ and $b$ be the probabilities of a 1 and a 6, respectively, on the first die, and let $c$ and $d$ be the corresponding probabilities on the second die.

Since $1/11$ is the probability of rolling a total of 2, $ac = 1/11$ so $c = 1/(11a)$; since $1/11$ is the probability of rolling 12, $bd = 1/11$ so $d = 1/(11b)$. Since the probability of rolling a 7 through the combination $1+6$ or $6+1$ is at most $1/11$,

$$\begin{aligned}\frac{1}{11} &\geq ad + bc \\ \frac{1}{11} &\geq \frac{a}{11b} + \frac{b}{11a} \\ 1 &\geq \frac{a}{b} + \frac{b}{a}.\end{aligned}$$

Since $a/b$ and $b/a$ are reciprocals, one of them is at least 1, so this inequality cannot hold.

6. Let $ABC$ be a triangle with $\angle A = 120°$. The bisector of $\angle A$ meets side $BC$ at $D$. Prove that

$$\frac{1}{AD} = \frac{1}{AB} + \frac{1}{AC}.$$

*Solution.* The area of $\triangle ABC$ is the sum of the areas of triangles $ABD$ and $ADC$, so

$$\begin{aligned}\frac{1}{2}AB \cdot AC \cdot \sin 120° &= \frac{1}{2}AB \cdot AD \cdot \sin 60° + \frac{1}{2}AD \cdot AC \cdot \sin 60° \\ \frac{1}{2}AB \cdot AC \cdot \frac{\sqrt{3}}{2} &= \frac{1}{2}AB \cdot AD \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}AD \cdot AC \cdot \frac{\sqrt{3}}{2} \\ AB \cdot AC &= AB \cdot AD + AD \cdot AC.\end{aligned}$$

Dividing through by $AB \cdot AC \cdot AD$ gives the desired result.

7. Let $n$ and $k$ be positive integers with $n < \sqrt{(k-1)2^k}$. Prove that it is possible to color each element of the set $\{1, 2, \ldots, n\}$ red or green such that no $k$-term arithmetic progression is monochromatic.

*Solution.* Let $A$ be the number of $k$-term arithmetic progressions in $\{1, 2, \ldots, n\}$. For any common difference $d$, an arithmetic progression of difference $d$ fits in $\{1, 2, \ldots, n\}$ iff its initial term $a$ satisfies

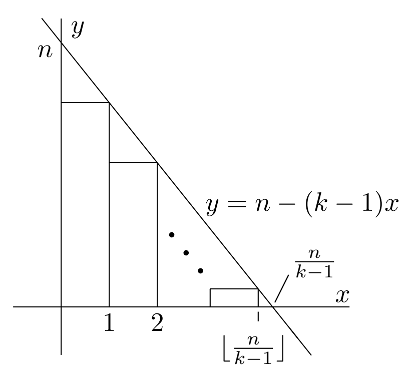
$$1 \leq a < a + (k-1)d \leq n$$

which is equivalent to

$$1 \leq a \leq n - (k-1)d;$$

this inequality has $n - (k-1)d$ solutions, as long as $n - (k-1)d \geq 0$. Thus the total number of arithmetic progressions is

$$A = \sum_{d=1}^{\lfloor \frac{n}{k-1} \rfloor} (n - (k-1)d).$$




As shown in the figure, the terms of this sum can be seen as the areas of nonoverlapping rectangles lying under the graph of $y = n - (k-1)x$. Their sum therefore does not exceed the area of the triangle enclosed by this line and the axes:

$$A \leq \frac{1}{2} \cdot \frac{n}{k-1} \cdot n = \frac{n^2}{2(k-1)}.$$

Now consider any $k$-term arithmetic progression. The number of colorings in which it is monochromatic is

$$2^n \cdot \frac{2}{2^k}$$

since, of the $2^k$ ways that its terms might be colored, only 2 are monochromatic. Therefore the number of colorings that make *no* $k$-term arithmetic progression monochromatic is at least

$$\begin{aligned}
&2^n - A\left(2^n \cdot \frac{2}{2^k}\right) \\
&\geq 2^n - \frac{n^2}{2(k-1)} \cdot 2^n \cdot \frac{2}{2^k} \\
&= 2^n \left(1 - \frac{n^2}{(k-1)2^k}\right).
\end{aligned}$$

If $n^2 < (k-1)2^k$ (equivalently, $n < \sqrt{(k-1)2^k}$) then this lower bound will be positive, implying that there is at least one coloring with the desired property.