Berkeley Math Circle
Monthly Contest 7
Due April 5, 2011

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 7, Problem 3
by Bart Simpson
in grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Let x and y be integers such that $\dfrac{3x+4y}{5}$ is an integer. Prove that $\dfrac{4x-3y}{5}$ is an integer.

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2. Six rooks are placed on a 6×6 chessboard, at the locations marked $+$, so that each rook "attacks" the five squares in the same row and the five squares in the same column. Determine if it is possible to label each empty square with a digit (0 through 9) so that for each rook, the ten squares which it attacks are all labeled with different digits.

3. Let $f(n)$ be the number of digits of a positive integer n (in base 10). Prove that
$$f(2^n) + f(5^n) = n + 1.$$

4. Let ABC be a triangle with incenter I. A line through I parallel to BC intersects sides AB and AC at D and E respectively. Prove that the perimeter of $\triangle ADE$ is equal to $AB + AC$.

5. Two dice are loaded so that the numbers 1 through 6 come up with various (possibly different) probabilities on each die. Is it possible that, when both dice are rolled, each of the possible totals 2 through 12 has an equal probability of occurring?

6. Let ABC be a triangle with $\angle A = 120^\circ$. The bisector of $\angle A$ meets side BC at D. Prove that
$$\frac{1}{AD} = \frac{1}{AB} + \frac{1}{AC}.$$

7. Let n and k be positive integers with $n < \sqrt{(k-1)2^k}$. Prove that it is possible to color each element of the set $\{1, 2, \ldots, n\}$ red or green such that no k-term arithmetic progression is monochromatic.