Berkeley Math Circle
Monthly Contest 6 – Solutions

1. Two thousand and eleven positive integers are chosen, all different and less than or equal to 4020. Prove that two of them have no common factors except 1.

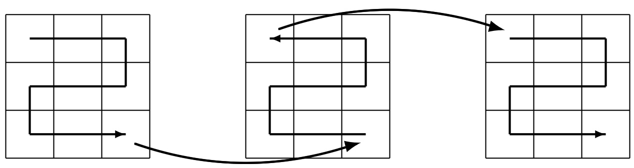
   Solution. Split the numbers from 1 to 4020 into the 2010 pairs $\{1,2\},\{3,4\},\ldots,\{4019,4020\}$. Since 2011 numbers are chosen but there are only 2010 pairs, two numbers have to lie in the same pair. These numbers cannot have any common factors except 1, because any divisor of both numbers would have to divide their difference, which is 1.

2. A $3\times3\times3$ cube is made out of 27 subcubes. On every face shared by two subcubes, there is a door allowing you to move from one cube to the other. Is it possible to visit every subcube exactly once if

   (a) You may start and end wherever you like

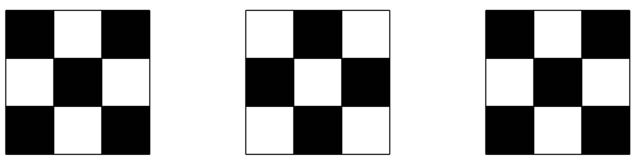
   (b) You must start at the center subcube?

   Solution. (a) It is possible. Here is one of many possible routes.



Level 1 Level 2 Level 3

   (b) It is impossible. Color the subcubes black and white alternately as shown:



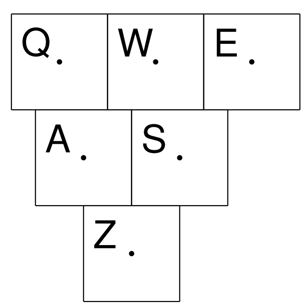
Level 1 Level 2 Level 3

   Every door connects a black subcube to a white subcube. Since the central subcube is white, the route must begin

$$\text{White} \to \text{Black} \to \text{White} \to \text{Black} \to \cdots.$$

   Examining the first 27 subcubes visited, we see that 14 are white and 13 are black, a contradiction since the actual cube has 14 black and 13 white subcubes.

3. In this fragment of a computer keyboard, the keys are congruent squares touching along their edges, and each letter refers to the point at the center of the corresponding key. Prove that triangles QAZ and ESZ have the same area.



   Solution. Let us use measuring units in which the side length of each key is 1. We express the area of quadrilateral $QAZE$ in two ways:

   (a) By dividing into triangles QAZ and QZE. Since $\triangle QZE$ has base $QE = 2$ and height 2, we get

$$\text{Area } QAZE = \text{Area } QAZ + \frac{1}{2}\cdot 2\cdot 2 = \text{Area } QAZ + 2.$$

(b) By dividing to triangles ESZ, QWA, WAS, WES, and ASZ. The four latter triangles all have base 1, height 1, and area $1/2$, so

$$\text{Area } QAZE = \text{Area } ESZ + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \text{Area } ESZ + 2.$$

Since quadrilateral $QAZE$ must have the same area in both computations, we deduce that Area $QAZ =$ Area ESZ.

4. Let x, y, z, and u be real numbers satisfying the equation

$$\frac{x-y}{x+y} + \frac{y-z}{y+z} + \frac{z-u}{z+u} + \frac{u-x}{u+x} = 0.$$

Suppose that x, y, and z are rational (i.e. each is the quotient of two integers) and distinct. Prove that u is rational as well.

Solution. Let us begin by combining the fractions in pairs.

$$\begin{aligned} 0 &= \left(\frac{x-y}{x+y} + \frac{y-z}{y+z}\right) + \left(\frac{z-u}{z+u} + \frac{u-x}{u+x}\right) \\ &= \frac{(x-y)(y+z)+(x+y)(y-z)}{(x+y)(y+z)} + \frac{(z-u)(u+x)+(z+u)(u-x)}{(z+u)(u+x)} \end{aligned}$$

After expanding the numerators and collecting like terms, we get

$$\begin{aligned} 0 &= \frac{2xy-2yz}{(x+y)(y+z)} + \frac{2zu-2ux}{(z+u)(u+x)} \\ &= 2(x-z)\left[\frac{y}{(x+y)(y+z)} - \frac{u}{(z+u)(u+x)}\right]. \end{aligned}$$

Because $x \neq z$, we can divide out $2(x-z)$ to get

$$\begin{aligned} 0 &= \frac{y}{(x+y)(y+z)} - \frac{u}{(x+u)(u+z)} \\ &= \frac{y(x+u)(u+z) - u(x+y)(y+z)}{(x+y)(y+z)(x+u)(u+z)}. \end{aligned}$$

We now expand the numerator and get, after canceling some terms,

$$\begin{aligned} 0 &= u^2y - uy^2 + xyz - uxz \\ &= uy(u-y) + xz(y-u). \end{aligned}$$

If $u = y$, then we are done since y is rational. Assuming $u \neq y$, we divide through by $(u-y)$ to get $0 = uy - xz$. Solving for u, we get $u = xz/y$ which is rational given that x, y, and z are.

5. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive real numbers such that for all $n \geq 1$,

$$a_n \leq a_{2n} + a_{2n+1}.$$

Prove that there exists an $N \geq 1$ such that

$$\sum_{n=1}^{N} a_n > 1.$$

Solution. Let us prove by induction that for any positive integer k, there is an N for which

$$\sum_{n=1}^{N} a_n \geq k a_1. \tag{1}$$

The base case, $k = 1$ and $N = 1$, is trivial. Given that (1) is true for a given k, we have

$$k a_1 \leq \sum_{n=1}^{N} a_n \leq \sum_{n=1}^{N} (a_{2n} + a_{2n+1}) = \sum_{\substack{2 \leq n \leq 2N \\ n \text{ even}}} a_n + \sum_{\substack{3 \leq n \leq 2N+1 \\ n \text{ odd}}} a_n = \sum_{n=2}^{2N+1} a_n.$$

Adding a_1, we get

$$\sum_{n=1}^{2N+1} a_n \geq (k+1)a_1$$

as desired.

The problem now follows by taking k large enough so that $k > 1/a_1$, so $ka_1 > 1$.

6. Determine whether there exists a 2011×2011 matrix with the following properties:

   - Every cell is filled with an integer from 1 to 4021.
   - For every integer i ($1 \leq i \leq 2011$), the ith row and the ith column together contain every integer from 1 to 4021.

   Solution. Answer: no. Fix an integer k from 1 to 4021. Let us say that an index i "hits" a cell containing the number k if the cell is in either the ith row or the ith column. The conditions stipulate that each index hits exactly one instance of k, so the total number of hits is 4021. On the other hand, every cell not lying on the main diagonal is hit by exactly two indices (its row number and its column number), while those on the diagonal are hit only once. In particular, to create an odd total number of hits, k must appear on the diagonal. This is a contradiction since there are 4021 permissible values of k and only 2011 spots on the diagonal.

7. A *lattice point* is a point in the coordinate plane both of whose coordinates are integers. In $\triangle ABC$, all three vertices are lattice points and the area of the triangle is $1/2$. Prove that the orthocenter of $\triangle ABC$ is also a lattice point.

   Solution. Let us position our coordinate system so that A is the origin. Let $B = (a, b)$ and $C = (c, d)$. Then the formula for the area of a triangle with given vertex coordinates gives $\frac{1}{2} = \frac{1}{2}|ad - bc|$, i.e. $ad - bc = \pm 1$.

   Let $H = (x, y)$ be the orthocenter of $\triangle ABC$. The condition $CH \perp AB$ can be expressed in vector form as

$$0 = \overrightarrow{AB} \cdot \overrightarrow{CH} = \langle a, b \rangle \cdot \langle x - c, y - d \rangle$$

   or

$$ax + by = ac + bd. \tag{2}$$

   Similarly the condition $BH \perp AC$ can be expressed as

$$cx + dy = ac + bd. \tag{3}$$

   Adding d times equation (2) to $-b$ times equation (3) causes y to cancel, leaving

$$(ad - bc)x = d(ac + bd) - b(ac + bd).$$

   Since $ad - bc = \pm 1$, it follows that x is an integer. Similarly, we can prove that y is an integer as well.