Berkeley Math Circle
Monthly Contest 6
Due March 8, 2011

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 6, Problem 3
by Bart Simpson
in grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

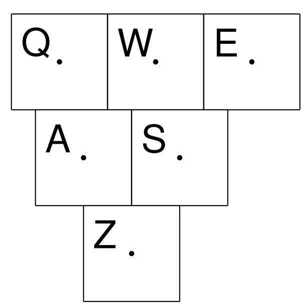
1. Two thousand and eleven positive integers are chosen, all different and less than or equal to 4020. Prove that two of them have no common factors except 1.

2. A $3 \times 3 \times 3$ cube is made out of 27 subcubes. On every face shared by two subcubes, there is a door allowing you to move from one cube to the other. Is it possible to visit every subcube exactly once if

   (a) You may start and end wherever you like

   (b) You must start at the center subcube?

   Remark. For each part (a and b) you must provide an answer (yes or no) and a proof that your answer is correct.



3. In this fragment of a computer keyboard, the keys are congruent squares touching along their edges, and each letter refers to the point at the center of the corresponding key. Prove that triangles QAZ and ESZ have the same area.

4. Let x, y, z, and u be real numbers satisfying the equation

$$\frac{x-y}{x+y} + \frac{y-z}{y+z} + \frac{z-u}{z+u} + \frac{u-x}{u+x} = 0.$$

   Suppose that x, y, and z are rational (i.e. each is the quotient of two integers) and distinct. Prove that u is rational as well.

5. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive real numbers such that for all $n \geq 1$,

$$a_n \leq a_{2n} + a_{2n+1}.$$

Prove that there exists an $N \geq 1$ such that

$$\sum_{n=1}^{N} a_n > 1.$$

6. Determine whether there exists a 2011×2011 matrix with the following properties:

    - Every cell is filled with an integer from 1 to 4021.
    - For every integer i ($1 \leq i \leq 2011$), the ith row and the ith column together contain every integer from 1 to 4021.

7. A *lattice point* is a point in the coordinate plane both of whose coordinates are integers. In $\triangle ABC$, all three vertices are lattice points and the area of the triangle is $1/2$. Prove that the orthocenter of $\triangle ABC$ is also a lattice point.