Berkeley Math Circle
Monthly Contest 4 – Solutions

1. For an arrangement of the digits 0 through 9 around a circle, a number is called a *neighbor sum* if it is the sum of some two adjacent digits in the arrangement. For example, the arrangement

	0	8	3	
4				5
7				9
	1	6	2	

has five neighbor sums: 4, 7, 8, 11, and 14. What is the minimal possible number of neighbor sums, given that each digit must be used just once?

Solution. The answer is 3. It can be achieved using the arrangement

	1	8	2	
9				7
0				3
	5	4	6	

in which the only neighbor sums are 5, 9, and 10. To prove that no smaller number is possible, first consider the digit 0. It is adjacent to two different numbers between 1 and 9 inclusive, yielding two neighbor sums in the range $1, \ldots, 9$. Now consider the digit 9. It is adjacent to two numbers and at least one of them is *not* 0. Thus 9 has a neighbor between 1 and 8 inclusive, and thus there is a neighbor sum in the range $10, \ldots, 17$. Since these ranges do not overlap, there are altogether at least three different neighbor sums in the range $1, \ldots, 17$.

2. A gadget has four dials in a row, each of which can be turned to point to one of three numbers: 0 (left), 1 (up) or 2 (right). Initially the dials are in the respective positions 2, 0, 1, 0, so that the gadget reads "2010." You may perform the following operation: choose two adjacent dials pointing at different numbers, and turn them to point to the third number. For example, taking the first two dials, you could change "2010" to "1110." Is it possible to perform a sequence of such operations so that the gadget reads "2011"?

Solution. The answer is no. We notice that initially the sum of the numbers on the dials is 3. We claim that after each operation, the sum of the numbers on the dials remains a multiple of 3. To see this, consider the three possible types of moves:

(a) Changing a 0 and a 2 to two 1's does not change the digit sum.

(b) Changing a 0 and a 1 to two 2's increases the sum by 3.

(c) Changing a 1 and a 2 to two 0's decreases the sum by 3.

Thus, the sum always goes up or down by multiples of 3, and thus we cannot reach the position 2011 in which the sum is 4.

3. Suppose that ABC and $A'B'C'$ are two triangles such that $\angle A = \angle A'$, $AB = A'B'$, and $BC = B'C'$. Suppose also that $\angle C = 90°$. Prove that triangles ABC and $A'B'C'$ are congruent.

Solution. Construct a point D on ray AC such that $AD = A'C'$. Then $\triangle ABD \cong \triangle A'B'C'$ by SAS. If $D = C$, then we are done, so assume that $D \neq C$. We have $BD = B'C'$ and also $BC = B'C'$, so right triangle BCD has hypotenuse BD equal to leg BC, which is a contradiction.

4. Evaluate the sum

$$\frac{1}{1+\tan 1^\circ}+\frac{1}{1+\tan 2^\circ}+\frac{1}{1+\tan 3^\circ}+\cdots+\frac{1}{1+\tan 89^\circ}.$$

(The *tangent* (tan) of an angle α is the ratio BC/AC in a right triangle ABC with $\angle C = 90^\circ$ and $\angle A = \alpha$, and its value does not depend on the triangle used.)

Solution. By examining a triangle with angles x, $90 - x$, and 90, it is not hard to see the trigonometric identity

$$\tan(90^\circ - x) = \frac{1}{\tan x}.$$

Let us pair up the terms

$$\frac{1}{1+\tan x} \quad \text{and} \quad \frac{1}{1+\tan(90^\circ - x)}$$

for $x = 1, 2, 3, \ldots, 44$. The sum of such a pair of terms is

$$\begin{aligned}\frac{1}{1+\tan x}+\frac{1}{1+\tan(90^\circ - x)} &= \frac{1}{1+\tan x}+\frac{1}{1+\frac{1}{\tan x}}\\ &= \frac{1}{1+\tan x}+\frac{\tan x}{\tan x + 1}\\ &= \frac{1+\tan x}{1+\tan x} = 1.\end{aligned}$$

There is one more term in the sum, the middle term:

$$\frac{1}{1+\tan 45^\circ} = \frac{1}{1+1} = \frac{1}{2}.$$

Therefore the sum is $44(1) + 1/2 = 89/2$.

5. Several positive integers are written on the blackboard. You can erase any two numbers and write their greatest common divisor (GCD) and least common multiple (LCM) instead. Prove that eventually the numbers will stop changing.

Solution. First of all, since

$$a \cdot b = \gcd(a, b) \cdot \operatorname{lcm}(a, b)$$

for all positive integers a and b, the product of the numbers on the board does not change.

Second, we claim that at each step in which the numbers change, their sum increases. Suppose we pick two numbers a and b ($a \le b$) and replace them with $g = \gcd(a, b)$ and $\ell = \operatorname{lcm}(a, b)$. If $g = a$, then $\ell = b$ and the numbers do not change, so $g < a$ and therefore $g < b$. The change in the sum of the numbers is

$$(g+\ell)-(a+b) = \frac{g^2+g\ell-ga-gb}{g} = \frac{g^2+ab-ga-gb}{g} = \frac{(a-g)(b-g)}{g} > 0.$$

Since each number on the blackboard cannot exceed the product of the numbers that were originally there, the sum cannot increase unboundedly, and thus the numbers must stop changing.

6. For all positive integers n, prove that

$$\sum_{k=1}^{n} \phi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{n(n+1)}{2}.$$

(For a positive integer n, $\phi(n)$ denotes the number of positive integers less than or equal to n and relatively prime to n. For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Solution. Consider the fractions a/b, where a and b range over integers such that $1 \le a \le b \le n$. We will count these fractions in two ways:

(a) By unreduced form. For each denominator b, there are b possible numerators $a = 1, 2, \ldots, b$, so the total number of fractions is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

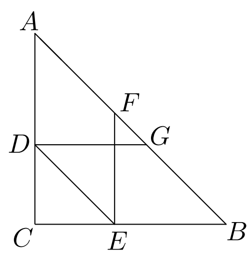
(b) By reduced form. Suppose a fraction a/b has been reduced to c/d. Given the denominator d, there are $\phi(d)$ possible numerators c such that $c \le d$ and c/d is in lowest terms. To get from c/d back to a/b, we must multiply numerator and denominator by a positive integer k such that $kc \le kd \le n$. Since $kc \le kd$ always holds if $c \le d$, the choice of k is limited only by the inequality $kd \le n$, which has $\lfloor n/d \rfloor$ solutions. Thus there are $\phi(d)\lfloor n/d \rfloor$ fractions a/b for a given choice of d, so the total number of fractions is

$$\sum_{d=1}^{n} \phi(d) \left\lfloor \frac{n}{d} \right\rfloor.$$

Since both methods of counting must yield the same answer, the identity follows.

7. We are given a right isosceles triangle with legs of length 1 inside which every point (including vertices, points on sides, and all points in the interior) is colored red, yellow, green, or blue. Prove that there are two points of the same color such that the distance between them is at least $2 - \sqrt{2}$.

Solution.



Define points D, E, F, G on the sides of the triangle as shown such that $AD = AF = GB = BE = 2 - \sqrt{2}$. Then $CD = CE = \sqrt{2} - 1$, and since $\triangle CDE$ is isosceles and right, $DE = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$. Now segments DE and AF are equal and parallel, so $ADEF$ is a parallelogram and $EF = AD = 2 - \sqrt{2}$. Similarly, $DG = 2 - \sqrt{2}$.

Assume that the conclusion is false, i.e. any two points at a distance of at least $2 - \sqrt{2}$ are different colors. Since $\triangle ABC$ has all side lengths greater than $2 - \sqrt{2}$, all of its vertices are different colors. Without loss of generality, assume that A is red, B is yellow, and C is green. Points F and G, which are at distances of at least $2 - \sqrt{2}$ from all three vertices, must both be blue. Point E, which is at a distance of at least $2 - \sqrt{2}$ from A, F, and B, must be green. Similarly, D is green. But D and E are a distance of $2 - \sqrt{2}$ from each other, so we have a contradiction.