Berkeley Math Circle
Monthly Contest 4
Due January 11, 2010

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, **and BMC level,** as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 4
by Bart Simpson
in grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. For an arrangement of the digits 0 through 9 around a circle, a number is called a *neighbor sum* if it is the sum of some two adjacent digits in the arrangement. For example, the arrangement

	0	8	3	
4				5
7				9
	1	6	2	

   has five neighbor sums: 4, 7, 8, 11, and 14. What is the minimal possible number of neighbor sums, given that each digit must be used just once?

   Remark. To receive the full 7 points, your solution must include the following:

   - An answer—the minimum number of neighbor sums.
   - An arrangement of the digits 0-9 which achieves this number of neighbor sums.
   - A proof of why no arrangement with fewer neighbor sums is possible.

2. A gadget has four dials in a row, each of which can be turned to point to one of three numbers: 0 (left), 1 (up) or 2 (right). Initially the dials are in the respective positions 2, 0, 1, 0, so that the gadget reads "2010." You may perform the following operation: choose two adjacent dials pointing at different numbers, and turn them to point to the third number. For example, taking the first two dials, you could change "2010" to "1110." Is it possible to perform a sequence of such operations so that the gadget reads "2011"? Prove rigorously your solution.

3. Suppose that ABC and $A'B'C'$ are two triangles such that $\angle A = \angle A'$, $AB = A'B'$, and $BC = B'C'$. Suppose also that $\angle C = 90°$. Prove that triangles ABC and $A'B'C'$ are congruent.

4. Evaluate the sum
$$\frac{1}{1+\tan 1°} + \frac{1}{1+\tan 2°} + \frac{1}{1+\tan 3°} + \cdots + \frac{1}{1+\tan 89°}.$$
   (The *tangent* (tan) of an angle α is the ratio BC/AC in a right triangle ABC with $\angle C = 90°$ and $\angle A = \alpha$, and its value does not depend on the triangle used.)

5. Several positive integers are written on the blackboard. You can erase any two numbers and write their greatest common divisor (GCD) and least common multiple (LCM) instead. Prove that eventually the numbers will stop changing.

6. For all positive integers n, prove that
$$\sum_{k=1}^{n} \phi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{n(n+1)}{2}.$$
(For a positive integer n, $\phi(n)$ denotes the number of positive integers less than or equal to n and relatively prime to n. For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

7. We are given a right isosceles triangle with legs of length 1 inside which every point (including vertices, points on sides, and all points in the interior) is colored red, yellow, green, or blue. Prove that there are two points of the same color such that the distance between them is at least $2 - \sqrt{2}$.