Berkeley Math Circle
Monthly Contest 3
Due December 7, 2010

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, **and BMC level,** as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 3
by Bart Simpson
in grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. You are given an $m \times n$ chocolate bar divided into 1×1 squares. You can break a piece of chocolate by splitting it into two pieces along a straight line that does not cut through any of the 1×1 squares. What is the minimum number of times you have to break the bar in order to separate all the 1×1 squares? Prove rigorously your answer.

2. Let n be a positive integer. Prove that the nth prime number is greater than or equal to $2n - 1$. (A prime number is a positive integer, such as $2, 3, 5, 7, \ldots$, which is not 1 and not divisible by any positive integer except itself and 1.)

3. Given the hypotenuse and the difference of the two legs of a right triangle, show how to reconstruct the triangle with ruler and compass.

   Remark. To receive the full 7 points, your solution must include the following:

   (a) A description (in words) that begins with the two segments, one representing the hypotenuse of the triangle and the other the difference of the sides, and ends up with a right triangle.

   (b) A proof that you have found the "right" triangle, that is, why no other right triangle can have the same hypotenuse and difference of legs.

4. Show that each number in the sequence

   $$49, \quad 4489, \quad 444889, \quad 44448889, \quad \ldots$$

   is a perfect square.

5. Let $\{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers such that for each $n \geq 1$,

   $$a_{n+2} = a_{n+1} + a_n.$$

   Prove that for all $n \geq 2$, the quantity

   $$|a_n^2 - a_{n-1}a_{n+1}|$$

   does not depend on n.

6. The inscribed circle of a triangle ABC touches the sides BC, CA, AB at D, E, and F respectively. Let X, Y, and Z be the incenters of triangles AEF, BFD, and CDE, respectively. Prove that DX, EY, and CZ meet at one point.

7. Define a sequence $a_0, a_1, a_2, \ldots$ in the following way: $a_0 = 0$, and for $n \geq 0$,

$$a_{n+1} = a_n + 5^{a_n}.$$

Let k be any positive integer. Prove that the remainders when $a_0, a_1, \ldots, a_{2^k-1}$ are divided by 2^k are all different.