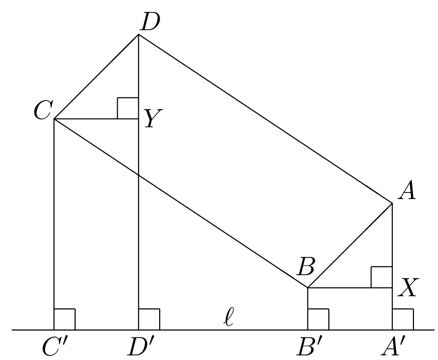
Berkeley Math Circle
Monthly Contest 2 – Solutions

1. Arrange the following number from smallest to largest: 2^{1000}, 3^{750}, 5^{500}.

   Solution. Since all the numbers are positive, taking the 250th root of each number will not change their ordering. The resulting numbers are $2^4 = 16$, $3^3 = 27$, and $5^2 = 25$. These have the ordering $2^4 < 5^2 < 3^3$, so the ordering of the original numbers is $2^{1000} < 5^{500} < 3^{750}$.

2. In the diagram at right, $ABCD$ is a parallelogram. Given that the distances from A, B, and C to line ℓ are, respectively, 3, 1, and 5, find the distance from D to ℓ and prove that your answer is correct.

   Remark. In the contest, the labels on the points A and C were switched. This makes the figure grossly out of scale but does not change the solution.



   Solution. As shown, label the feet of the perpendiculars from A, B, C, D to ℓ by A', B', C', D'. Also, drop perpendiculars BX and CY to AA' and DD' respectively. We find that triangles ABX and DCY are similar since their corresponding sides are parallel, and in fact congruent since AB and CD are opposite sides of the parallelogram. Thus $AX = DY$. But $AX = AA' - XA' = AA' - BB' = 3 - 1 = 2$, and hence $DD' = DY + YD' = AB + CC' = 2 + 5 = 7$.

3. Ten cups lie upside down in a line. It is known that pennies lie under two of the cups which are consecutive in the line. Choosing several of the cups, you may ask for the total number of coins under them. Is it possible to determine the positions of the pennies by asking two such questions, without knowing the answer to the first question before making the second?

   Solution. The answer is yes; here is one strategy that works. On the first turn choose the cups 1, 4, 5, 6, 7, and on the second turn choose 1, 6, 7, 8, 9. Depending on the locations of the coins, the answers will be:

Coins	$1,4,5,6,7$	$1,6,7,8,9$
$1,2$	1	1
$2,3$	0	0
$3,4$	1	0
$4,5$	2	0
$5,6$	2	1
$6,7$	2	2
$7,8$	1	2
$8,9$	0	2
$9,10$	0	1

   Since each possible location for the coins gives a different pair of answers, the answers are enough to reconstruct the location of the coins.

4. The sequences $\{x_n\}$ and $\{y_n\}$ are defined by $x_0 = 2$, $y_0 = 1$ and, for $n \geq 0$,

$$x_{n+1} = x_n^2 + y_n^2 \quad \text{and} \quad y_{n+1} = 2x_n y_n.$$

   Find and prove an explicit formula for x_n in terms of n.

   Solution. If we add the formulas for x_{n+1} and y_{n+1} together, we get

$$x_{n+1} + y_{n+1} = x_n^2 + y_n^2 + 2x_n y_n = (x_n + y_n)^2.$$

   Thus, increasing n by 1 raises $x_n + y_n$ to the power 2. Since $x_0 + y_0 = 3$, we get

$$x_n + y_n = (\cdots(3^2)^2 \cdots)^2 \text{ [n squarings]} = 3^{2^n}. \quad (1)$$

Similarly, we can subtract the formulas for x_{n+1} and y_{n+1} to get

$$x_{n+1} - y_{n+1} = x_n^2 + y_n^2 - 2x_n y_n = (x_n - y_n)^2.$$

However, $x_0 - y_0 = 1$. Since $1^2 = 1$, we get

$$x_n - y_n = 1 \tag{2}$$

for all n.

Finally, we add together equations (1) and (2), getting

$$2x_n = 3^{2^n} + 1$$

which yields an explicit formula

$$x_n = \frac{3^{2^n} + 1}{2}.$$

5. Let $ABCD$ be a square. Consider four circles k_1, k_2, k_3, k_4 which pass respectively through A and B, B and C, C and D, D and A, and whose centers are outside the square. Circles k_4 and k_1, k_1 and k_2, k_2 and k_3, k_3 and k_4 intersect respectively at L, M, N, P inside the square. Prove that quadrilateral $LMNP$ can be inscribed in a circle.

Solution. We use a standard theorem that states that a convex quadrilateral $WXYZ$ can be inscribed in a circle if and only if $\angle W + \angle Y = 180$. We have

$$\begin{aligned}
\angle PLM + \angle MNP &= (360 - \angle MLA - \angle ALP) + (360 - \angle PNC - \angle CNM) \\
&= (180 - \angle MLA) + (180 - \angle ALP) + (180 - \angle PNC) + (180 - \angle CNM) \\
&= \angle ABM + \angle PDA + \angle CDP + \angle MBC \\
&= (\angle ABM + \angle MBC) + (\angle PDA + \angle CDP) \\
&= 90 + 90 = 180,
\end{aligned}$$

as desired.

Remark. This problem works for any inscribed quadrilateral $ABCD$. It was restricted to squares to avoid the annoying issues of multiple diagram configurations and nonconvex quadrilaterals.

6. Seven people are sitting around a circular table, not necessarily equally spaced from each other. Several vases are standing on the table. We say that two people can see each other if there are no vases on the line segment connecting them. (Treat both people and vases as points.)

   (a) If there are 12 vases, prove that there are two people who can see each other.

   (b) If there are 13 vases, prove that one can rearrange the people and the vases in such a way that no two people can see each other.

Solution. (a) Assign each of the 21 line segments connecting two people a "weight" as follows: each side of the heptagon (AB, BC, CD, etc.) gets a weight of 1; each "skip-one" diagonal (AC, BD, etc.) gets a weight of $1/2$; each "skip-two" diagonal (AD, BE, etc.) gets a weight of $1/3$.

Claim 1. *The total weight of all the lines of sight (segments connecting two people) blocked off by any vase is at most* 1.

Proof. If the vase does not lie on any line of sight, the total weight is clearly 0. If the vase lies on a side, the total weight is 1 because the sides do not intersect any other sides or diagonals. If the vase lies on a skip-one diagonal, say AC, any other diagonal through the vase must have B as an endpoint, so there is at most one such diagonal and the total weight is at most $1/2 + 1/2 = 1$. Finally, if the vase lies on a skip-two diagonal, say AD, but not on any skip-one diagonal, any other diagonal passing through the vase is also a skip-two diagonal. There can be at most two such diagonals (since they must have an endpoint at B or C), so the total weight is at most $1/3 + 1/3 + 1/3 = 1$. □

We now see that the total weight of all the lines of sight is $7 + 7/2 + 7/3 = 12\frac{5}{6}$, so it is impossible to block them all off with 12 vases.

(b) Let six people sit at the vertices of a regular hexagon $ABCDEF$, and let the seventh person G sit on the arc AF such that CG passes through the intersection of AE and BF. Place one vase at the center of the table (hence on AD, BE, and CF) and one at the intersection of AE, BF, and CG. Then place vases on the intersections of: AC and BD; CE and DG; DF and EG; AF and BG. Finally, put one vase at the midpoint of each side of heptagon $ABCDEFG$. It is easy to see that no two people can see each other.

7. Let x, y, and z be positive integers satisfying $xy = z^2 + 1$. Prove that there are integers a, b, c, and d such that $x = a^2 + b^2$, $y = c^2 + d^2$, and $z = ac + bd$.

Solution. We will use strong induction on z. Assume first that $z = 1$. Then $xy = 1^2 + 1 = 2$, so x and y are 1 and 2 in some order. If $x = 2$ and $y = 1$, then there is a solution $a = b = c = 1$, $d = 0$; if $x = 1$ and $y = 2$, we may take $a = c = d = 1$, $b = 0$.

Now assume that $z > 1$ and that, whenever $Z < z$, the equation $XY = Z^2 + 1$ implies that there are integers A, B, C, and D such that $X = A^2 + B^2$, $Y = C^2 + D^2$, and $Z = AC + BD$. We note that neither x nor y can equal z, for then xy and z^2 would be two multiples of z although their difference is 1. We note that x and y cannot be both greater or both less than z since

$$(z-1)^2 < z^2 + 1 < (z+1)^2.$$

Thus, because the symmetry of the problem allows us to switch x and y, we can assume that $x > z > y$.

Let $X = x + y - 2z$, $Y = y$, and $Z = z - y$. Multiplying out shows that the equation $XY = Z^2 + 1$ is satisfied. Also, $Z < z$, so we have integers A, B, C, and D such that the following equations hold:

$$X = A^2 + B^2 \tag{3}$$
$$Y = C^2 + D^2 \tag{4}$$
$$Z = AC + BD \tag{5}$$

Let $a = A + C$, $b = B + D$, $c = C$, and $d = D$. Equation (4) already tells us that $y = c^2 + d^2$. Since $Y + Z = y + z - y = z$, we can add equation (4) and (5) together to get

$$z = Y + Z = AC + C^2 + BD + D^2 = (A + C)C + (B + D)D = ac + bd.$$

Finally, since $X + Y + 2Z = x + y - 2z + y + 2z - 2y = x$, we add (3), (4), and twice (5) to get

$$x = X + 2Z + Y = A^2 + 2AC + C^2 + B^2 + 2BD + D^2 = (A + C)^2 + (B + D)^2 = a^2 + b^2.$$

This completes the induction.