Berkeley Math Circle
Monthly Contest 2
Due November 2, 2010

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 2
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

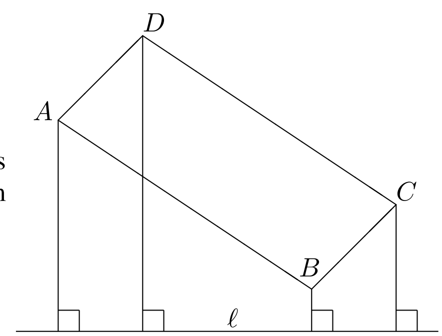
Enjoy solving these problems and good luck!

Problems

1. Arrange the following numbers from smallest to largest: 2^{1000}, 3^{750}, 5^{500}. Prove rigorously that your answer is correct.

   Remark. Solutions to this problem based on extensive calculator or computer calculations will not be accepted.

2. In the diagram at right, $ABCD$ is a parallelogram. Given that the distances from A, B, and C to line ℓ are, respectively, 3, 1, and 5, find the distance from D to ℓ and prove that your answer is correct.



3. Ten cups lie upside down in a line. It is known that pennies lie under two of the cups which are consecutive in the line. Choosing several of the cups, you may ask for the total number of coins under them. Is it possible to determine the positions of the pennies by asking two such questions, without knowing the answer to the first question before making the second? Explain rigorously your solution.

4. The sequences $\{x_n\}$ and $\{y_n\}$ are defined by $x_0 = 2$, $y_0 = 1$ and, for $n \geq 0$,

$$x_{n+1} = x_n^2 + y_n^2 \quad \text{and} \quad y_{n+1} = 2x_n y_n.$$

   Find and prove an explicit formula for x_n in terms of n.

5. Let $ABCD$ be a square. Consider four circles k_1, k_2, k_3, k_4 which pass respectively through A and B, B and C, C and D, D and A, and whose centers are outside the square. Circles k_4 and k_1, k_1 and k_2, k_2 and k_3, k_3 and k_4 intersect respectively at L, M, N, P inside the square. Prove that quadrilateral $LMNP$ can be inscribed in a circle.

6. Seven people are sitting around a circular table, not necessarily equally spaced from each other. Several vases are standing on the table. We say that two people can see each other if there are no vases on the line segment connecting them. (Treat both people and vases as points.)

   (a) If there are 12 vases, prove that there are two people who can see each other.

   (b) If there are 13 vases, prove that one can rearrange the people and the vases in such a way that no two people can see each other.

7. Let x, y, and z be positive integers satisfying $xy = z^2 + 1$. Prove that there are integers a, b, c, and d such that $x = a^2 + b^2$, $y = c^2 + d^2$, and $z = ac + bd$.