Berkeley Math Circle
Monthly Contest 1 – Solutions

1. Are there natural numbers a and b that make the equation

$$2a^2 + 1 = 4b^2$$

true? (The natural numbers are the counting numbers, $1, 2, 3, \ldots$.)

Solution. The answer is no. Suppose that a and b are any natural numbers. Then $2a^2$ is even, and so is $4b^2$. But an even number plus 1 will be odd, not even, so the equation can never be made to work, i.e. there are no natural numbers that satisfy the equation.

2. Given three points A, B, C known to lie on a circle, prove that one can reconstruct the original circle with a straightedge and compass.

Solution. Here is a suitable procedure: First draw circles centered at A and B with radius AB and join their two points of intersection to create the perpendicular bisector of AB. This line contains all points that have the same distance from A and B, so the center of a circle through A and B lies on it. Then, similarly construct the perpendicular bisector of AC. Because A, B, and C are *known* to lie on a circle, the two perpendicular bisectors are not parallel (or else no point could have the same distance from A, B, and C), so they must meet at a point O. Draw a circle centered at O with radius OA.

Since O lies on both perpendicular bisectors, this circle passes through B and C. Finally, any other circle through the same three points would have to have its center on both perpendicular bisectors, and hence at O. Its radius must equal OA, implying that it coincides with the constructed circle.

3. (a) You are given the expression

$$1 \diamond 2 \diamond 3 \diamond 4 \diamond 5 \diamond 6 \diamond 7.$$

Determine whether it is possible to replace one of the symbols $\diamond$ with $=$ and the other symbols $\diamond$ with $+$ or $-$ so as to end up with a correct equation.

Solution. It is possible; one of many solutions is

$$1 = 2 + 3 + 4 + 5 - 6 - 7.$$

(b) The same question for the expression

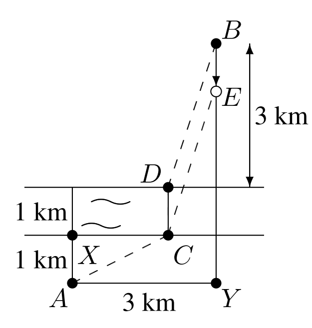
$$1 \diamond 2 \diamond 3 \diamond 4 \diamond 5 \diamond 6.$$

Solution. It is impossible. Suppose for the sake of contradiction that we have a solution. Change the $=$ sign to a $+$. We are now adding together two equal integers, so the sum must be even. Now change each $-$ sign, one by one, to a $+$. At each such step, we increase the value of the expression by twice the value of the number after the changed sign. This keeps the total even. But at the end of all these operations, we have changed all the signs to $+$, so the result is

$$1 + 2 + 3 + 4 + 5 + 6$$

which is 21, an odd number. Therefore, our assumption is false; i.e. there is no way to place the signs as required.

4. Two villages, A and B, lie on opposite sides of a straight river in the positions shown. There will be a market in village B, and residents of village A wish to attend. The people of village A would like to build a bridge across and perpendicular to the river so that the total route walked by residents of A to the bridge, across the bridge, and onward to B is as short as possible. How can they find the exact location of the bridge, and how long will the route be?



Solution. For convenience, we label the point X, the foot of the perpendicular from A to the river. We also label Y, the point where the perpendicular BY to the river through B meets the parallel AY to the river through A. The key construction is to translate B southward 1 km to point E. Let $ACDB$ be any route. Then segments DC and BE are congruent and parallel, making a parallelogram $DCEB$. The total route is $AC + CD + DB = AC + CE + EB$. But by the triangle inequality, $AC + CE \geq AE$ so the total route cannot be shorter than $AE + EB = 5 + 1 = 6$ km. *(continued)*

The residents of A can ensure a 6-kilometer route by putting the start C of the bridge at the point where AE crosses the lower bank of the river. In this situation, since A, C, and E lie on a line, the triangle inequality is an equality, so the total route is $AC + CD + DB = AC + CE + EB = AE + EB = 5 + 1 = 6$.

Remark. Using the fact that triangles ACX and EAY are similar if A, C, and E are collinear, it is possible to calculate that the optimal location of point C is 0.75 km east of X.

5. We have four bowls labeled A, B, C, D in a row, and we have four indistinguishable marbles which we can initially distribute among the bowls any way we like. A *move* consists of transferring one marble from a bowl to one of the adjacent bowls in the row. Is it possible to perform a succession of moves in which every distribution of the marbles appears exactly once?

Solution. The answer is no. Call a position of the marbles "even" if bowls A and C have an even total number of marbles and "odd" otherwise. It is easy to see that any move changes an even position to an odd position and vice versa. However, note that, for any nonnegative integer k, the number of ways to distribute k marbles among two bowls is $k+1$ (as the first bowl can hold any number of marbles from 0 to k). Therefore it is easy to count the number of even and odd positions:

Marbles in A and C	Marbles in B and D	Results
4, in 5 ways	0, in 1 way	$5 \cdot 1 = 5$ even positions
3, in 4 ways	1, in 2 ways	$4 \cdot 2 = 8$ odd positions
2, in 3 ways	2, in 3 ways	$3 \cdot 3 = 9$ even positions
1, in 2 ways	3, in 4 ways	$2 \cdot 4 = 8$ odd positions
0, in 1 way	4, in 5 ways	$1 \cdot 5 = 5$ even positions

The totals are 19 even positions and 16 odd. It is not possible to visit all 19 even positions without the 18 odd positions separating them containing some duplicates.

6. In the interior of a triangle ABC with area 1, points D, E, and F are chosen such that D is the midpoint of AE, E is the midpoint of BF, and F is the midpoint of CD. Find the area of triangle DEF.

Solution. Let x be the area of $\triangle DEF$. Comparing triangles ABE and DEF, we find that base AE is twice base DE but, since E bisects BF, the heights to these bases are equal. Thus $\triangle ABE$ has area $2x$. Symmetrically, triangles BCF and CAD have area $2x$. Since these four triangles fill up $\triangle ABC$, we have $1 = x + 2x + 2x + 2x = 7x$, so $x = 1/7$.

7. Let a, b, and c be positive real numbers such that

$$abc = 1$$

and

$$a + b + c > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Prove that for every positive integer n,

$$a^n + b^n + c^n > \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n}.$$

Using $abc = 1$, we transform the given condition to

$$a + b + c > bc + ca + ab$$

or

$$-bc - ca - ab + a + b + c > 0.$$

We then add $abc - 1$ ($= 0$) to the left side, getting

$$abc - bc - ca - ab + a + b + c - 1 > 0$$

which factors as

$$(a-1)(b-1)(c-1) > 0. \quad (1)$$

In exactly the same way we transform the condition to be proved to

$$(a^n - 1)(b^n - 1)(c^n - 1) > 0. \quad (2)$$

However, for any positive real x, the numbers $x - 1$ and $x^n - 1$ are both positive, both negative, or both zero. Consequently (1) and (2) are equivalent.