LINKING

(BERKELEY MATH CIRCLE: 2 FEBRUARY 2010)

1. Some basic notions from knot theory

Knot: smooth, simple closed curve in 3-space

Link: disjoint collection of knots (the components of the link)†

Examples (diagrams will be drawn on the board):

(1) unknot, trefoil, figure eight, ...
real-world examples: orbit of the earth; flight path of a bothersome bee

(2) 2-component unlink, Hopf link, Whitehead link, ...
real-world examples: pair of orbits, e.g. earth and a comet; intertwining strands of circular DNA

(3) 3-component unlink, Borromean rings, ...

Every link L has an associated mirror image $\overline{L}$ (you get a diagram for $\overline{L}$ by changing all the crossings in a diagram for L). A link is achiral if it is isotopic to its mirror image, i.e. if $L = \overline{L}$.

Problem 1 Show that the figure eight knot and the Hopf link are achiral. What about the trefoil knot, the Whitehead link and the Borromean rings?

Now it turns out that every link is the boundary of a surface in 3-space; indeed there are infinitely many such surfaces. (For today, we will assume that all our surfaces are "connected" in the sense that any two points on the surface can be joined by a path in the surface.)

Problem 2 Sketch some surfaces bounded by the unknot, the trefoil, and the Hopf link.

We will be particularly interested in oriented links and surfaces. What does this mean?

An oriented link is a link together with a direction assigned to each component, indicated by arrows in the diagram.

An oriented surface is a surface with one of its two sides designated "up" and the other "down", indicated by drawing an upward arrow at some point on the surface. But beware: not all surfaces can be oriented! Only the 2-sided ones can, and of course there exist 1-sided surfaces such as the Mobius band.

†Two links are considered the same if you can find an isotopy (meaning a motion of space) that carres one onto the other. On the diagramatic level, any such isotopy can be broken down into a sequence of Reidemeister moves. These notions were discussed with Maia Averett in your 12/1/09 BMC session.

Note that an orientation on a surface F induces an orientation on each boundary curve C of F (directed away from you if you are standing on the up-side of F near C, with C on your right). Write ∂F for this oriented boundary. The orientation also gives a sign to each transverse intersection point x of an oriented curve J with F, equal to $+1$ or -1 according to whether J is directed up or down at x. The sum of these signs is called the intersection number of J with F, denoted $J \bullet F$.

Theorem (Seifert 1934) *Every oriented link L bounds an oriented surface F in 3-space.*
(*Any such surface is called a Seifert surface for L.*)

Proof sketch (Seifert's algorithm) Resolve a diagram of L to get the Seifert circles, which bound a disjoint collection of disks. Now join these disks by half-twisted bands at the crossings to get F. □

Problem 3 Show that F is oriented, with $\partial F = L$.

2. The linking number

Goal: Find a way to measure the intertwining of the components J and K of an oriented 2-component link L. For example, you can count the number of times J wraps around K, in the following sense:

Geometric Definition The linking number of J with K is[†] the intersection number of J with any Seifert surface F of K,

$$\mathrm{lk}(J,K) \ := \ J \bullet F \ \in \ \mathbb{Z}$$

(The proof that this integer is the same for all choices of F requires more advanced techniques) **Examples** ...

There is also a combinatorial definition for the linking number in terms of the set $C_J^K(D)$ of crossings of J *under* K in any given diagram D for L:

$$\mathrm{lk}(J,K) \ = \ \textstyle\sum_{\times \in C_J^K(D)} \mathrm{sign}(\times).$$

Here sign($\times$) is $+1$ or -1 according to whether the upper strand passes from left to right or vice versa as you approach $\times$ along the lower strand. (Draw pictures of crossings of each sign so that you become comfortable with this.) **Examples** ...

[†]Historically, the first definition for the linking number of J with K was given by Gauss in 1833, growing out of his study of the intertwining of the earth's orbit with the orbits of other celestial objects. It takes the form of a double integral:

$$\mathrm{lk}(J,K) \ = \ \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{2\pi} \frac{dx}{ds} \times \frac{dy}{dt} \bullet \frac{x-y}{\|x-y\|^3}\, ds\, dt$$

where $x = x(s)$ and $y = y(t)$ parametrize the components of L. You can of course ignore this if you don't have a multivariable calculus background. Otherwise, try to figure out why this might measure intertwining (think about what the triple product $u \times v \bullet w$ of vectors measures, geometrically). The fact that this definition coincides with the one above is an exercise in advanced calculus.

Problem 4 Show (using the Reidemeister moves) that the combinatorial definition yields the same value for all diagrams D of L. Thus, although less intuitive than the geometric definition, it is easier to show well-defined.

Problem 5 Use the combinatorial definition to show that the linking number is symmetric, meaning

$$\mathrm{lk}(J, K) = \mathrm{lk}(K, J).$$

(Hint: rotate by 180° about an axis in the projection plane) Thus we can write $\mathrm{lk}(L)$ for this common value when $L = J \cup K$, without ordering the components. Also show the linking number is chiral, meaning that $\mathrm{lk}(\overline{L}) = -\mathrm{lk}(L)$.

Problem 6 Show that the combinatorial definition follows from the geometric one applied to the surface obtained from D by Seifert's algorithm.

Examples Special oriented links T_n for $n \in \mathbb{Z}$ with $\mathrm{lk}(T_n) = n$ (pictures will be drawn on the board). In particular $T_0 =$ unlink U and $T_{\pm 1} =$ oriented Hopf links H and $\overline{H}$. Reversing one component of T_n and then taking the mirror image gives another oriented link $\overline{T}_n$ of linking number n that turns out to be different from T_n for $n \neq 0$, and important for our considerations below (draw pictures).

Note that the linking number of either Whitehead link, W or $\overline{W}$ (with any chosen orientation), is zero, just like the unlink! This suggests the

Question Are W, $\overline{W}$ and the unlink really distinct?

To answer this question, note that any 2-component link can be converted into one of the T_n's by a link homotopy, meaning a deformation in which individual components may cross themselves, but distinct components may not intersect. In fact:

Key Lemma *If D and D' are diagrams of oriented links with the same linking numbers, then there is a sequence of Reidemeister moves and self-crossing changes* (*i.e. between a component and itself*) *that converts D into D'.*

Problem 7 Prove this lemma.

This is tricky but instructive: To start, explain how to convert the diagram for the first component into a diagram of the unknot by changing some of its crossings (ignoring the second component).

The key lemma provides the basis for a recursive definition of an "enhanced" linking number that is in many cases more discerning than the standard linking number.

3. The enhanced linking number

Let $L_+ = J \cup K$ be an oriented 2-component link that has a diagram D with a preferred positive self-crossing of J. Suppose that near this crossing, D looks like  , where the first two strands come from J and the last one comes from K.

A local change in D, appearing as  near the preferred crossing, produces another link which we call L_-. A further local change to  produces a 3-component link L_0 in which J has split into two components. Combining either of these with K produces two 2-component links L and R (for "left" and "right") which locally appear as  or  . The ordered 4-tuple of links (L_+, L_-, L, R) will be called a skein quadruple.

Theorem (Livingston 2003) *Each 2-component oriented link L can be assigned an integer $\lambda(L)$, its enhanced linking number, in a unique way so that*

(1) (*crossing changes*) *For any skein quadruple (L_+, L_-, L, R),*

$$\lambda(L_+) - \lambda(L_-) \;=\; \mathrm{lk}(L)\mathrm{lk}(R).$$

(2) (*normalization*) *$\lambda(T_n) = 0$ for all $n \in \mathbb{Z}$.*

The uniqueness of λ follows from the key lemma (do you see how?) but the existence is deeper. It relies on a certain polynomial invariant of knots called the Conway polynomial which we will not discuss here.

Instead let's see how to use this enhanced linking number to answer our previous question about the Whitehead links, and more generally to explore chirality questions.

Problem 8 Compute $\lambda(W)$ and $\lambda(\overline{W})$, and conclude that W, $\overline{W}$ and the unlink are distinct. In particular W is chiral (see Prob. 1).

Problem 9 Show that $\lambda(\overline{T}_n) = (n - n^3)/6$, and conclude that T_n (forgetting the orientation) is achiral if and only if $n = 0, \pm 1$.

Problem 10 * Show that there exist oriented achiral 2-component links with any prescribed odd linking number, but that there do not exist any such links whose linking number n is twice an odd number. (Hint for the latter†: show that if L is such a link, then $\lambda(L) - \lambda(T_n) = \lambda(\overline{T}_n) - \lambda(\bar{L})$ (for a suitable orientation on $\bar{L}$) and use the formula for $\lambda(\overline{T}_n)$ from the last problem.

Problem 11 ** Find an achiral 2-component link of linking number 8, or show one doesn't exist

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†which is a theorem due to Kirk and Livingston, 1997