USAMO 2010 PREPARATION SESSION
APRIL 6TH, 2010

Problem 1: Let ABC be a triangle such that $\angle A = 2\angle B$. Prove that $a^2 = b(b+c)$, where a, b, c are length of sides BC, AC, AB respectively.
Solution Sketch: Let D be a point on BC such that $\angle BAD = \angle B$. Then, $|BD| = |AD|$ and triangles ABC and DAC are similar. The equality follows immediately.

Problem 2: Find all integers m, n such that

$$3 \cdot 2^m + 1 = n^2$$

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Solution Sketch: Rewrite the equation as $3 \cdot 2^m = (n-1)(n+1)$. Then, there are integers p and q such that $3 \cdot 2^p = n-1$, $2^q = n+1$ or $3 \cdot 2^p = n+1$, $2^q = n-1$. Or equivalently $3 \cdot 2^p - 2^q = \pm 2$. At this point, looking at a few cases, we can find that only $(m,n) = (0,2) = (3,5) = (4,7)$ satisfy the equation.

Problem 3: For non-negative integers $a < b$ let $M(a,b)$ be the arithmetic mean of $\sqrt{i^2+3i+3}$ for $a \leq i \leq b$. Compute the whole part of $M(a,b)$ as a closed form function of a and b.
Solution Sketch: It is easy to show that

$$i + 3/2 < \sqrt{i^2+3i+3} < i+2$$

Then, summing these inequalities, we get

$$\frac{a+b}{2} + \frac{3}{2} < M(a,b) < \frac{a+b}{2} + 2$$

Thus, if a and b have the same parity, then $[M(a,b)] = \frac{a+b+2}{2}$. Otherwise, $[M(a,b)] = \frac{a+b+3}{2}$

Problem 4: Find all functions $f : R^+ \Rightarrow R^+$ such that

$$(1 + yf(x))(1 - yf(x+y)) = 1$$

for all $x, y \in R^+$, where R^+ is a set of all positive real numbers.
Solution Sketch: Let $g(x) = 1/f(x)$. Then, the equation becomes $g(x+y) - g(x) = y$. Let $h(x) = g(x) - x$. Then, the equation becomes $h(x+y) = h(x)$. It is obvious that only constant function can satisfy this. It is easy to check that $f(x) = 1/(x+c)$ for any non-negative c satisfies then condition.

Problem 5: In subset A of set $1, 2, \ldots, 2010$, the difference of any two elements is not a prime number. What is the maximum number of elements that can be in A.
Solution Sketch: Let $A = a_1, a_2, \ldots, a_k$ and let $a_1 < a_2 < \ldots < a_k$. Furthermore, divide A into groups of consecutive integers and call these groups clusters. It is obvious that a cluster can contain either 1 or 2 elements. In the former case, if a_i is a cluster, then the next cluster can start no earlier than with $a_i + 4$. We say that a_i "ate" 3 numbers ($a_i + 1$, $a_i + 2$, $a_i + 3$ cannot appear in A). In the latter case, if $a_i, a_i + 1$ is a cluster, then, the next cluster can begin no earlier than with $a_i + 9$. In this case, we say that a_i ate $a_i + 2$, $a_i + 3$, $a_i + 4$ and $a_i + 1$ ate $a_i + 5$, $a_i + 6$, $a_i + 7$, $a_i + 8$. Thus, in any case, a number in A eats at least three numbers after itself. It is obvious that no number is eaten twice.

Thus, summing the numbers in A and the eaten numbers, we can write the following inequality

$$k + 3(k-2) \leq 2010$$

We have $k-2$ because the last cluster can have two elements and can be at the very end not eating any numbers lower than 2011. Thus, we have that $k \leq 504$. However, if $k = 504$, the last cluster has to have two elements, it has to be at the very end and all other clusters have to have one element and be spaced exactly 3 numbers between each other. This case, does not work, because 2005 and 2010 has to be in A but their difference is prime. Thus, $k \leq 503$, which is achievable with a set $1, 5, 9, \ldots, 2009$.

Problem 6: Consider $n > 2$ checkers placed at the vertices of a regular n-gon. Each checker is colored red on one side and blue on another side. In one move, one can choose any three consecutive checkers and flip them up side down. A "situation" is any configuration of checkers (situations that are different only by a rotation, are considered different). How many situations can be reached from a given one with a finite number of moves?

Solution Sketch: For $n = 3k + 1$ and $n = 3k + 2$, it is easy to find a set of moves that change the color of only one checker. Thus, all 2^n situations are reachable from any situation.

If $n = 3k$, let $s_0 = c_3, c_6, \ldots, c_{3k}$, $s_1 = c_1, c_4, \ldots, c_{3k-2}$, $s_2 = c_2, c_5, \ldots, c_{3k-1}$, where c_i's are checkers. Let n_i be the number of top-red checkers in s_i. There are 8 possible cases of parities of n_is. We will shows that given a situation, we can reach all situations in the same case and all situations in the "opposite" case (where the parities of all n_is are changed). Thus, the total number of situations reachable when $n = 3k$, is $2^n/8 + 2^n/8 = 2^{n-2}$.

First, it is obvious that we cannot reach more than this, because given an initial situation, each more changes the case to the opposite one and we can never land into any one of the other 6 cases.

To show that we can reach all the situations in these two cases, we just need to see that with two moves that overlap on two checkers we can change the colors of two checkers in one of s_is without changing anything else. After that, the argument is pretty simple.

Problem 7: Let a, b, and c be non-negative real numbers and x, y, and z be positive real numbers such that $a + b + c = x + y + z$. Prove that

$$\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} \geq a + b + c$$

Solution Sketch: We just need to show that

$$\frac{a^3}{x^2} + \frac{b^3}{y^2} \geq \frac{(a+b)^3}{(x+y)^2}$$

and apply it twice. This inequality is pretty easy. All one needs to do is to simplify it, let $w = x/y$ and $d = a/c$, divide both sides by b^3y^4 so that the inequality becomes in terms of w and d. Then, one can factor out $(w-d)^2$ to get the final obvious inequality of

$$(w - d)^2(2wd + w^2 + 2w + d) \geq 0$$

Problem 8: Let O be the center of inscribed circle of an acute triangle ABC. Let points A_0, B_0, C_0 be the centers of circumscribed circles around triangles BCO, ACO, ABO respectively. Prove that lines AA_0, BB_0, CC_0 intersect in one point.

Solution Sketch: It is not hard to see that

$$\angle BA_0C_0 = \angle C_0A_0O = \angle OA_0B_0 = \angle B_0A_0C \equiv \alpha$$
$$\angle CB_0A_0 = \angle A_0B_0O = \angle OB_0C_0 = \angle C_0B_0A \equiv \beta$$
$$\angle AC_0B_0 = \angle B_0C_0O = \angle OC_0A_0 = \angle A_0C_0B \equiv \gamma$$

Let A_1 be the intersection of B_0C_0 and AA_0. Define B_1 and C_1 analogously. Then,

$$\frac{|B_0A_1|}{|A_1C_0|} = \frac{S_{AA_0B_0}}{S_{AA_0C_0}} = \frac{|AB_0||A_0B_0|\sin 3\beta}{|AC_0||A_0C_0|\sin 3\gamma} = \frac{\sin\gamma|A_0B_0|\sin 3\beta}{\sin\beta|A_0C_0|\sin 3\gamma}$$

Writing down analogous expressions for $\frac{|C_0B_1|}{|B_1A_0|}$ and $\frac{|A_0C_1|}{|C_1B_0|}$ and multiplying all of them we get 1. By Ceva's theorem, we are done.

Problem 9: Prove that for any natural number k, there exists a natural number n such that n has exactly k different prime factors and $2^{n^2} + 1$ is divisible by n^3.

Problem 10: Let I be the center of circle w inscribed in trapezoid $ABCD$. Sides AD and BC intersect at point R. Let P and Q be the tangent points of W with sides AB and CD, respectively. Let the line passing through P and perpendicular to PR intersect lines AI and BI at points A_0 and B_0, respectively. Also, let the line passing through Q and perpendicular to QR intersect lines CI and DI at points C_0 and D_0, respectively. Prove that $A_0D_0 = B_0C_0$.