High School
Test 2

I. Multiple-Choice Questions – There is only one right answer to each of the following 10 questions. Please write down the corresponding letter in the provided space on the left of each question.

____ 1. Given a sequence $\{a_n\}$ which is defined by the formula $a_n = n + \frac{\lambda}{n}$, $n = 1, 2, 3, \ldots$. If $\{a_n\}$ is a monotonic increasing sequence, then what is the range of value for λ?

(A) $(-\infty, 2)$ (B) $(-\infty, 1)$ (C) $(-1, +\infty)$ (D) $(-2, +\infty)$

____ 2. A real valued function $f(x)$ satisfies the following conditions:

(1) $f(x-3) = f(3-x)$ for any x;

(2) $f(x+2) = f(2-x)$ for any x;

(3) $f(x) = \ln x$ when $x \in [2,4]$.

Then one of the following choices is the correct order for $f(-7)$, $f(-4)$, $f(-2)$.

(A) $f(-7) < f(-4) < f(-2)$ (B) $f(-2) < f(-4) < f(-7)$

(C) $f(-2) < f(-7) < f(-4)$ (D) $f(-7) < f(-2) < f(-4)$

____ 3. If $2\sin^2\theta + \sqrt{3}\sin\theta\cos\theta - 3\cos^2\theta = 0$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, then what is the value for $\sin^3 2\theta + \cos^3 2\theta$?

(A) $\frac{3\sqrt{3}-1}{8}$ (B) $\frac{1-3\sqrt{3}}{8}$ (C) $-\frac{3\sqrt{3}+1}{8}$ (D) $\frac{1+3\sqrt{3}}{8}$

____ 4. If $|x-1| + |x-2| + |x-3| + \cdots + |x-2008| \geq m$ for any real number x, then what is the maximum possible value for m ?

(A) 1003×1004 (B) 1004^2 (C) 1003×1005 (D) 1004×1005

____ 5. Let α be the angles between two adjacent sides of regular octagonal cones. What range of value does α take on for all possible regular octagonal cones ?

(A) $\left(\frac{\pi}{2}, \pi\right)$ (B) $\left(\frac{3\pi}{4}, \pi\right)$ (C) $\left(\frac{7\pi}{8}, \pi\right)$ (D) $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

____ 6. Let A and B be two points on the ellipse $\frac{x^2}{3} + y^2 = 1$ and $AB \cdot \boldsymbol{e} = 0$, where $\boldsymbol{e}$ is an unit vector on the y-axis. Then, as the area of ΔAOB reaches its maximum, what is $\angle AOB$?

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{2\pi}{3}$ (D) $\pi - 2\arctan 3$

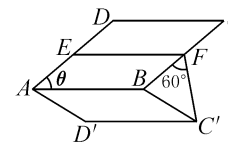
____ 7. For $a > 0$, if inequality $\sqrt{x} > ax + \frac{1}{2}$ has the solution set (b, 4), then what is the value for $a + b$?

(A) $\frac{19}{8}$ (B) $\frac{35}{8}$ (C) $\frac{25}{24}$ (D) $\frac{59}{72}$

____ 8. For $a > 1$, if α and β are the two roots for the equation $a^x |\log_a x| = 1$, then which one of the following choices represents the relationship between $\alpha\beta$ and 1 ?

(A) $\alpha\beta > 1$ (B) $\alpha\beta = 1$ (C) $\alpha\beta < 1$ (D) Non-determined

____ 9. Given a parallelogram $ABCD$ with $\angle BAD = \theta$ (θ is an acute angle) as in Figure 1. Let E and F be the midpoints on AD and BC, respectively. Along EF, fold the parallelogram $EFCD$ so that C' and D' are the new positions for C and D, respectively, and $\angle BFC = \frac{\pi}{3}$. If the area of $ABFE$ is twice the area of $ABC'D'$, then what is the angle measurement of θ ?



(A) $\frac{\pi}{3}$ (B) $\arcsin\sqrt{\frac{2}{5}}$ (C) $\arcsin\sqrt{\frac{3}{5}}$ (D) $\arcsin\sqrt{\frac{4}{5}}$

____ 10. Let F be the right focus point for the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and ΔABC is a inscribed triangle inside the ellipse so that $\vec{FA} + \vec{FB} + \vec{FC} = \vec{0}$. Let d_1, d_2, and d_3 be the distance from A, B, and C to the right directrix, respectively. Then what is the value for $d_1 + d_2 + d_3$?

(A) 6 (B) 8 (C) 12 (D) 16

II. Fill In the Blank Questions – Write the appropriate answer to fill in the blank space provided.

11. The number of solution for $\log x = \sin x + 1$ is ______.

12. If $2\cos\alpha + 3\sin\beta = 4$, then the minimum value for $3\cos\alpha + 4\sin\beta$ is ______.

13. An even function $f(x)$ has [–2, 2] as its domain and it is monotonically increasing in the interval [–2, 0]. If $f(1-m) > f(m)$, then m will takes on the range of value for _______.

14. In a coordinate system, a lattice point is a point where both the x- and y-coordinates are integers.

Then the number of lattice points that satisfy $1 \le (|x|-1)^2 + (|y|-2)^2 \le 4$ is ______.

15. If three letters are put into three empty mailboxes A, B, and C in a random way, then the probability that there is only one letter in mailbox A is ______ and the probability that there is at least one letter in mailbox A is ______.

16. Let a, b, c be nonzero real numbers. If $a\cos x + b\sin x = c$ has two distinct roots α and β in the interval $(0, \pi)$, then $\alpha + \beta =$ ______.

17. Let R and O be the radius and center, respectively, of the circumscribed sphere of tetrahedron (regular triangular pyramid) P-ABC, O_1 be the centroid of equilateral ΔABC, and $PO = \frac{2}{3} PO_1$.

Then the arc length on the sphere between points A and B is _______. (Express your answer in terms of arccos)

18. Translate the hyperbola $F_1 : \frac{x^2}{16} - \frac{y^2}{9} = 1$ along the vector $\boldsymbol{a} = (-1, 3)$ to derived a new hyperbola F_2. Now, reflect F_2 using $y = x - 2$ as the axis of symmetry to derived another hyperbola F_3. Then the coordinates for the two foci of F_3 are _________.

19. In ΔABC, $\angle A = 90^\circ$, $BC = 1$, and PQ is a line segment with A as its midpoint and length d. If we rotate PQ using A as the center, then the minimum value for $BP \bullet CQ =$ ______. At that point, the included angle between PQ and BC is ______°.

20. Given that points $A(4, 0)$ and $B(3, 3)$. The minimum value for $|PA| + |PB|$ is ______, if P is any point on the ellipse $\frac{x^2}{36} + \frac{y^2}{20} = 1$.

III. Word Problems – Show all calculation.

21. Given a right square pyramid S-$ABCD$ with the square length of a and top vertex angle of 30°

for each of its 4 triangles.
(1) Find the planar angle that is formed by two adjacent sides of *S-ABCD*;
(2) Let *M* be the midpoint of *SA*. If you move a point starting from *M* along the 4 sides of $S\text{-}ABCD$ (but not passing through the vertex *S*) and return to *M*, then what is the minimum length of all such paths?

22. Given a real valued function $f(x) = \frac{1}{2}x^3 + \frac{3}{2}x$ and a sequence $\{a_n\}$ defined as $a_1 = b$ and $a_{n+1} = f(a_n)$ where $b \in (0,1)$ and $(n \in \mathbf{N}^*)$.
(1) Determine whether $\{a_n\}$ is monotonically increasing or decreasing. Prove your answer.
(2) Is there a real number *c* so that $0 < \frac{a_n + c}{a_n - c} < 2$ for all $n \in \mathbf{N}^*$. If such a *c* exists, find all possible values. If not, prove why not.

23. Given A and B are pairs of points on a parabola $y^2 = 2px\ (p > 0)$ that satisfy $OA \bullet OB = 0$ where *O* is the origin. For each pair *A* and *B*, let *M* be a point on the line segment *AB* such that $OM \bullet AB = 0$.
(1) Find the formula for locus of all such *M*;
(2) If $|OM| = \sqrt{3}p$, find the angle of elevation of *AB* (the angle of *AB* makes with the positive *x*-axis). Also, find the formula for the line *AB*.

2008 High School
Test 2 Solutions

1. A
2. C
3. C
4. B
5. B
6. C
7. D
8. C
9. D
10. D
11. 31
12. $-\frac{5}{2}$
13. $\left(\frac{1}{2}, 2\right]$
14. 36
15. $\frac{3}{10}; \frac{3}{5}$
16. $2\operatorname{arccot}\frac{a}{b}$
17. $\left(\pi - \arccos\frac{1}{8}\right)R$
18. (5, 2) or (5, –8)
19. $-\frac{d^2}{4} - \frac{d}{2}; 180^\circ$
20. $12 - 5\sqrt{8}$
21. (1) $\pi - \arccos\left(-7+4\sqrt{3}\right)$ (2) $\frac{3\sqrt{2}+\sqrt{6}}{4}a$
22. (1) Monotonically increasing sequence (2) $0 < c < \frac{b}{3}$
23. (1) $(x-p)^2 + y^2 = p^2, (x, y) \neq (0, 0)$ (2) Angle $= \frac{\pi}{3}$, $y = \sqrt{3}(x - 2p)$

I. Multiple-Choice Questions – There is only one right answer to each of the following 10 questions.

Please write down the corresponding letter in the provided space on the left of each question.

1. Since $a_{n+1} > a_n$ for any *n*, $n+1+\frac{\lambda}{n+1} > n+\frac{\lambda}{n}$. Solving for λ, $\lambda < n(n+1)$. However, $n \ge 1$. So $\lambda < 2$. (A)
2. Condition (1) implies $f(x) = f((x+3)-3) = f(-(x+3)+3) = f(-x)$. So, $y = f(x)$ is an even function. Condition (2) implies $f(x+4) = f(2+(x+2)) = f(2-(x+2)) = f(-x) = f(x)$. So, $y = f(x)$ is a periodic function with a period of 4. Therefore, $f(-7) = f(-7+8) = f(1) = f(2-1) = f(2+1) = f(3) = \ln 3$, $f(-4) = f(4) = \ln 4$, $f(-2) = f(2) = \ln 2$. (C)

3. $2\sin^2\theta + \sqrt{3}\sin\theta\cos\theta - 3\cos^2\theta = 0 \Rightarrow 2\tan^2\theta + \sqrt{3}\tan\theta - 3 = 0$ or $\tan\theta = \frac{-\sqrt{3} \pm 3\sqrt{3}}{4}$. Because $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $\tan\theta = -\sqrt{3}$ or $\theta = \frac{2\pi}{3}$. $\sin^3\theta - \cos^3 2\theta = \frac{3\sqrt{3}+1}{8}$. (C)

4. $|x-1|+|x-2|+|x-3|+\ldots+|x-2008|$
$=(|x-1|+|x-2008|)+(|x-2|+|x-2007|)+\ldots+(|x-1004|+|x-1005|)$
$\ge |(x-1)-(x-2008)|+|(x-2)-(x-2007)|+\ldots+|(x-1004)-(x-1005)|$
$=2007+2005+\ldots+3+1=1004^2$
(B)

5. As in Figure 2, let *P-ABCDEFGH* be a regular octagonal cone and *O* is the center of the base. When $P \to O$ along the line *OP*, two adjacent faces collapse to one plane. So, $\alpha \to \pi$. When $P \to \infty$, the regular octagonal cone becomes a regular octagon cylinder with ∞ height and $\alpha \to \frac{8-2}{8}\pi = \frac{3}{4}\pi$. Therefore, $\alpha \in \left(\frac{3\pi}{4}, \pi\right)$.

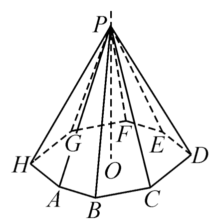


Figure 2

(B)

6. Because $\overrightarrow{AB} \cdot \vec{e} = 0$, $AB \parallel x$-axis. We can suppose point $B(x_0, y_0)$ is in the First Quadrant and $x_0 = \sqrt{3}\cos\theta$ and $y_0 = \sin\theta$, $0 < \theta < \frac{\pi}{2}$. Then the area of $\triangle AOB$ is $S = x_0 y_0 = \sqrt{3}\cos\theta\sin\theta = \frac{\sqrt{3}}{2}\sin 2\theta$. When $\theta = \frac{\pi}{4}$, *S* obtains its maximum value at $S_{\max} = \frac{\sqrt{3}}{2}$. And $\tan BOx = \frac{y_0}{x_0} = \frac{\sin\theta}{\sqrt{3}\cos\theta} = \frac{\sqrt{3}}{3}\tan\theta = \frac{\sqrt{3}}{3}\tan\frac{\pi}{4} = \frac{\sqrt{3}}{3}$. Therefore, $\angle BOx = \frac{\pi}{6}$, $\angle AOB = \pi - 2\times\frac{\pi}{6} = \frac{2\pi}{3}$. (C)

7. See Figure 3. Suppose $x = 4$ is a solution for $\sqrt{x} = ax + \frac{1}{2}$. Then $a = \frac{3}{8}$. And if $x = b$ is also a solution, then $\sqrt{b} = \frac{3}{8}b + \frac{1}{2}$ or $b = \frac{4}{9}$. Therefore, $a + b = \frac{59}{72}$. (D)

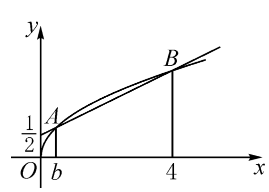
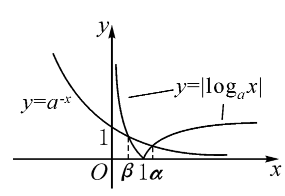


Figure 3



8. See Figure 4. For $a > 1$, the solution for $\log_a x = a^{-x}$ is greater than 1 and the solution for $-\log_a x = a^{-x}$ is less than 1. Therefore, $\alpha > 1$ and $\beta < 1$. And $0 > a^{-\alpha} - a^{-\beta} = \log_a \alpha + \log_a \beta = \log_a \alpha\beta$. So $\alpha\beta < 1$. (C)

Figure 4

9. See Figure 5. Find H on DC so that $BH \perp CD$ and BH intersects EF at G. Bend the parallelogram $EFCD$ so that H to H' and $\angle BFC' = \frac{\pi}{3} = 60^\circ$. Then $C'H' = CH$. Connect BH' and GH'. Since $BG \perp EF$ and $GH' \perp EF$ and $EF \perp$ plane BGH', so $EF \perp BH'$ and $C'D' \perp BH'$. Suppose $BC = 2a$. Then $CH = C'H' = 2a\cos\theta$. Also, $\triangle BC'F$ is an equilateral triangle so $BC' = a$.

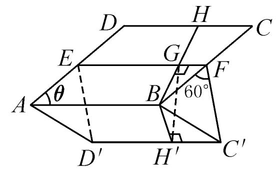


Figure 5

$BH'^2 = BC'^2 - C'H'^2 = a^2 - (2a\cos\theta)^2 = a^2(1 - 4\cos^2\theta)$ $(0 < \theta < 180^\circ)$.

Since we know $S_{ABFE} = 2S_{A'BC'D'}$, so we have $BG = 2BH'$ and $BG^2 = 4BH'^2$

or $a^2\sin^2\theta = 4a^2(1 - 4\cos^2\theta)$ or $\theta = \arcsin\sqrt{\frac{4}{5}}$. (D)

10. For this ellipse, $a = 5$, $b = 4$, $c = 3$, $e = \frac{3}{5}$, $F = (3, 0)$. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be points on the ellipse such that $\overrightarrow{FA} + \overrightarrow{FB} + \overrightarrow{FC} = \vec{0}$. Then $(3 - x_1) + (3 - x_2) + (3 - x_3) = 0$ or $x_1 + x_2 + x_3 = 9$. It is known that $|FA| = a - ex_1$, $|FB| = a - ex_2$, $|FC| = a - ex_3$. So, $|FA| + |FB| + |FC| = 3a - e(x_1 + x_2 + x_3)$. Since $d_1 = \frac{|FA|}{e}$, $d_2 = \frac{|FB|}{e}$, $d_3 = \frac{|FC|}{e}$, $d_1 + d_2 + d_3 = \frac{1}{e}(|FA| + |FB| + |FC|)$ $= \frac{3a}{e} - (x_1 + x_2 + x_3) = \frac{3 \times 5}{\frac{3}{5}} - 9 = 16$. (D)

11. Because $-1 \le \sin x \le 1$, $0 \le \sin x + 1 \le 2$ and $0 \le \lg x \le 2$, $0 \le x \le 100$.
From Figure 6, we know that:

(1) When $0 \le x \le \frac{3\pi}{2}$ these two curves have only one intersection which means the equation $\lg x = \sin x + 1$ has one solution.

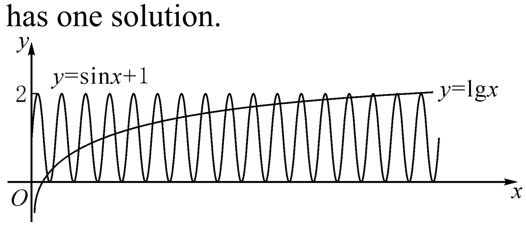


Figure 6

(2) When $\frac{3\pi}{2} < x \le 30\pi + \frac{3\pi}{2}$, there is a total of 15 cycles and these two curves have 2 intersections within each cycle which means the equation $\lg x = \sin x + 1$ has $2 \times 15 = 30$ solutions.

(3) Because $3\times 2\pi - \frac{\pi}{2} < 10 < 3\times 2\pi$, so when $3\times 2\pi - \frac{\pi}{2} < x \le 10$, these two curves have no intersection which means the equation $\lg x = \sin x$ has no solution.

Therefore, the equation $\lg x = \sin x$ has 31 solutions.

12. Because $2\cos\alpha + 3\sin\beta = 4$, $\cos\alpha = 2 - \frac{3}{2}\sin\beta \in \left[\frac{1}{2}, 1\right]$ and $\sin\beta = \frac{4 - 2\cos\alpha}{3} \in \left[\frac{2}{3}, 1\right]$. Also, $\sin\beta$ has a maximum value of 1 when $\cos\alpha$ reaches a minimum value of $\frac{1}{2}$ and $\sin\beta$ has a minimum value of $\frac{2}{3}$ when $\cos\alpha$ reaches a maximum value of 1. So, $3\cos\alpha - 4\sin\beta$ has a minimum value of $3\times\frac{1}{2} - 4\times 1 = -\frac{5}{2}$.

13. According to the problem, the possible values for m should satisfy $\begin{cases} -2 \le 1-m \le 2, \\ -2 \le m \le 2 \end{cases}$. Solving for m and we have $-1 \le m \le 2$.

(1) When $-1 \le m < 0$, $1 < 1 - m \le 2$. Since $f(x)$ is an even function and $-2 < m - 1 < -1$, $f(1-m) = f(m-1) < f(m)$ which contradicts the condition in the problem. So m cannot take on this range of values.

(2) When $0 \le m \le 1$, $f(x)$ is monotonically decreasing and $0 \le 1 - m \le 1$. So, $1 - m < m$ and $m > \frac{1}{2}$.

(3) When $1 < m \le 2$, $-1 \le 1 - m < 0$ and $-2 \le -m < -1$. Since $f(x)$ is an even function and monotonically increasing, $f(1-m) > f(-m) = f(m)$. So $1 < m \le 2$ satisfies the condition.

Therefore, m will take on the range of values of $\frac{1}{2} < m \le 2$.

14. Since the problem involves $|x|$ and $|y|$, the indicated region is symmetric with respect to the x-axis and y-axis and origin. From Figure 8, points (1, 1), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), and (3, 2) in the First Quadrant satisfy the problem. Therefore, there are 28 points in all four quadrants that satisfy the problem. In addition, there are 8 more points (± 1, 0), (0, ± 1), (0, ± 2), (0, ± 3) on the two axes also satisfy the problem. So the total number of lattice points is 28 + 8 = 36.

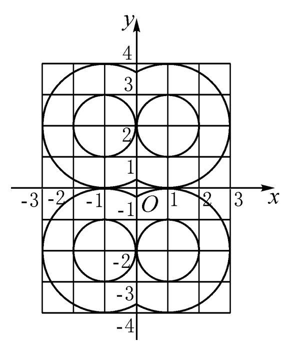


Figure 8

15. When Mailbox A has no letters, then three letters can be put into Mailboxes B and C in 4 ways: (3, 0), (2, 1), (1, 2), and (0, 3).

When mailbox A has 1 letter, then the other 2 letters can be put into Mailboxes B and C in 3 ways: (2, 0), (1, 1), and (0, 2).

When Mailbox A has 2 letters, then the other letter can be put into Mailboxes B and C in 2 ways: (1, 0), (0, 1).

When Mailbox A has all three letters, then there is only one way for Mailboxes B and C: (0, 0). Therefore, there is a total of 10 ways to distribute these 3 letters and the probability for Mailbox A having one letter is $\frac{3}{10}$ and the probability of Mailbox A having at least one letter is $\frac{3}{10}+\frac{2}{10}+\frac{1}{10}=\frac{6}{10}=\frac{3}{5}$.

16. From the problem, $\begin{cases} a\cos\alpha + b\sin\alpha = c \\ a\cos\beta + b\sin\beta = c \end{cases}$. Subtracting the second equation from the first yields $a(\cos\alpha-\cos\beta)+b(\sin\alpha-\sin\beta)=0$. Using the Difference Identities, $-2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}+2b\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}=0$ or $\sin\frac{\alpha-\beta}{2}\left(b\cos\frac{\alpha+\beta}{2}-a\sin\frac{\alpha+\beta}{2}\right)=0$. Because $\alpha\in(0,\pi)$ and $\beta\in(0,\pi)$ and $\alpha\neq\beta$, $\sin\frac{\alpha-\beta}{2}\neq 0$. So, $b\cos\frac{\alpha+\beta}{2}=a\sin\frac{\alpha+\beta}{2}$ or $\cot\frac{\alpha+\beta}{2}=\frac{a}{b}$. Since $0<\frac{\alpha+\beta}{2}<\pi$, $\frac{\alpha+\beta}{2}=\operatorname{arccot}\frac{a}{b}$ or $\alpha+\beta=2\operatorname{arccot}\frac{a}{b}$.

17. As in Figure 9, connect OA and O_1A. In the right triangle OO_1A, $OA = R$ and $OO_1=\frac{R}{2}$. So, $O_1A=\frac{\sqrt{3}}{2}R$ and $AB=2O_1A\cos 30^\circ=\frac{3}{2}R$. Connect OB to form an isosceles triangle $\triangle AOB$, $\cos\angle AOB=\frac{OA^2+OB^2-AB^2}{2OA\cdot OB}=-\frac{1}{8}$. So, $\angle AOB=\arccos\left(-\frac{1}{8}\right)=\pi-\arccos\frac{1}{8}$ and the arc length on the sphere between points A and B is $\left(\pi-\arccos\frac{1}{8}\right)R$.

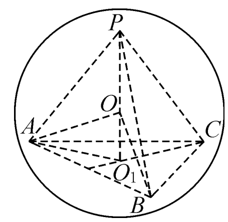
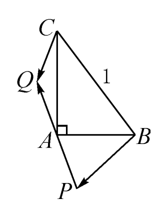


Figure 9

18. After the translation the resulting equation for F_2 is $\frac{(x+1)^2}{16}-\frac{(y-3)^2}{9}=1$ and the equation for F_3 is $\frac{(y+2+1)^2}{16}-\frac{(x-2-3)^2}{9}=1$ or $\frac{(y+3)^2}{16}-\frac{(x-5)^2}{9}=1$. So, $a^2=16$, $b^2=9$, and $c = 5$ and the coordinates for the two foci of F_3 are (5, 2) and (5, –8).

19. As in Figure 10, let θ be the angle between $\overrightarrow{PQ}$ and $\overrightarrow{BC}$, $0\le\theta\le 180^\circ$. Since $\angle BAC = 90^\circ$, $\overrightarrow{AB}\perp\overrightarrow{AC}$ or $\overrightarrow{AB}\cdot\overrightarrow{AC}=0$. Also, $\overrightarrow{AP}=-\overrightarrow{AQ}=-\frac{1}{2}\overrightarrow{PQ}$, $\overrightarrow{PB}=\overrightarrow{AB}-\overrightarrow{AP}$, $\overrightarrow{CQ}=\overrightarrow{AQ}-\overrightarrow{AC}$, and $\overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}$. So, $\overrightarrow{BP}\cdot\overrightarrow{CQ}=(\overrightarrow{AP}-\overrightarrow{AB})\cdot(\overrightarrow{AQ}-\overrightarrow{AC})$



$$= |\overrightarrow{AP}|^2 - \overrightarrow{AP}\cdot\overrightarrow{AC} - \overrightarrow{AB}\cdot\overrightarrow{AQ} + \overrightarrow{AB}\cdot\overrightarrow{AC}$$

$$= |\overrightarrow{AP}|^2 - \overrightarrow{AP}\cdot\overrightarrow{AC} + \overrightarrow{AB}\cdot\overrightarrow{AP}$$

$$= |\overrightarrow{AP}|^2 - \overrightarrow{AP}\cdot(\overrightarrow{AC} - \overrightarrow{AB})$$

Figure 10

$$= \left(\frac{d}{2}\right)^2 + \frac{1}{2}\overrightarrow{PQ}\cdot\overrightarrow{BC} = -\frac{d^2}{4} + \frac{d}{2}\cos\theta .$$

Therefore, when $\cos\theta = -1$ or $\theta = 180^\circ$ (PQ and BC are in opposite directions), $BP\cdot CQ$ has its minimum value of $-\frac{d^2}{4} + \frac{d}{2}$.

Figure 10

20. For this ellipse, $a^2 = 36$, $b^2 = 20$ and $c^2 = a^2 - b^2 = 16$. So, A (4, 0) is the right focus as shown in Figure 11. Let its left focus be $A'(-4,0)$. Connect $A'B$ and extend it to intersects with the ellipse at P', then $|PA| + |PB| \geq |P'A| + |P'B|$. From the definition of ellipse, we have $|PA| = 2a - |PA'|$. So, $|PA| + |PB| = 2a - |PA'| + |PB| = 2a - (|PA'| - |PB|) \geq 2a - |A'B|$ with equality holds when A', B, and P are collinear. Therefore, $(|PA| + |PB|)_{\min} = 2a - |A'B| = 12 - \sqrt{8}$.

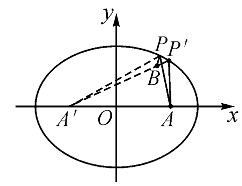


Figure 11

21. (1) As in Figure 12, $AB = a$, $\angle ASB = 30^\circ$. Find a point E on BC so that $BE \perp SC$. Then $DE \perp SC$.

Calculate $SB = \frac{a}{2\sin 15^\circ} = \frac{a}{2\sqrt{\frac{1-\cos 30^\circ}{2}}} = \frac{a}{\sqrt{2\left(1-\frac{\sqrt{3}}{2}\right)}} = \frac{a}{\sqrt{2-\sqrt{3}}} = \sqrt{2+\sqrt{3}}$.

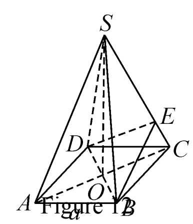


Figure 12

$DE = BE = \frac{1}{2}SB = \frac{\sqrt{2+\sqrt{3}}}{2}a$, $BD = \sqrt{2}a$. Use Laws of Cosines, $\cos\angle BED = \frac{BE^2 + DE^2 - BD^2}{2BE\cdot DE} = 1 - \frac{4}{2+\sqrt{3}} = 4\sqrt{3} - 7 < 0$. Therefore, $\angle BED = \pi - \arccos(7 - 4\sqrt{3})$.

(2) Cut this right pyramid along the edge SA and spread the figure into a planar figure as in Figure 13. Because $\angle MSM' = 30^\circ \times 4 = 120^\circ$, $MM'^2 = SM^2 + SM'^2 - 2SM\cdot SM'\cos 120^\circ = 2SM^2\left(1+\frac{1}{2}\right) = 3SM^2$. $MM' = \sqrt{3}SM = \sqrt{3}\frac{SA}{2} = \frac{\sqrt{3}}{2}SB = \frac{\sqrt{3}}{2}\times a\sqrt{2+\sqrt{3}} = \frac{3\sqrt{2}+\sqrt{6}}{4}$. Of course, the shortest distance between M and M' is the line segment MM'. Therefore, the length of the minimum path from M, along the 4 sides of $S\text{-}ABCD$ (but not passing through the vertex S) and return to M is $\frac{3\sqrt{2}+\sqrt{6}}{4}a$.

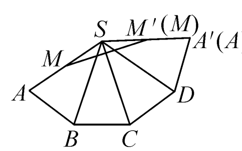


Figure 13

22. (1) The derivative of f, $f'(x) = -\frac{3}{2}x^2+\frac{3}{2} = \frac{3}{2}(1-x^2) > 0$ when $x\in(0,1)$. So f is monotonically increasing in (0, 1). $f(0) = 0$ and $f(1) = 1$ and $a_1\in(0,1)$. So, $0 = f(0) < f(a_1) = a_2 < f(1) = 1$. Similarly, $0 < a_3 < 1$, $\cdots$ Therefore, $a_{n+1} - a_n = f(a_n) - a_n = -\frac{1}{2}a_n^3+\frac{3}{2}a_n - a_n = -\frac{1}{2}a_n^3+\frac{1}{2}a_n$ $= \frac{1}{2}a_n(1-a_n^2) > 0$ is monotonically increasing.

(2) Supposed there is a positive real number c such that $0<\frac{a_n+c}{a_n-c}<2$ for all $n\in\mathbf{N}^*$. Then $\begin{cases}\frac{a_n+c}{a_n-c}>0,\\ \frac{3c-a_n}{a_n-c}<0,\end{cases}$ $(c>0,\ 0<a_n<1)$. Solving for c and we have $0<c<\frac{a_n}{3}$ for all $n\in\mathbf{N}^*$. Since $\{a_n\}$ is monotonically increasing, $\frac{a_n}{3}\geq\frac{a_1}{3}=\frac{b}{3}$. Therefore, $0<\frac{a_n+c}{a_n-c}<2$ when $0<c<\frac{a_n}{3}$.

23. (1) Because $\vec{OA}\cdot\vec{OB}=0$, $OA\perp OB$. Let the equation for the line OA be $y=kx\ (k\neq0)$. Then the equation for OB is $y=-\frac{1}{k}x$. From $\begin{cases}y^2=2px,\\ y=kx,\end{cases}$ we have $A\left(\frac{2p}{k^2},\frac{2p}{k}\right)$. Similarly, $B(2pk^2,-2pk)$.

(a) When the slope for line AB is defined which means when $\frac{2p}{k^2}\neq2pk^2$, $k\neq\pm1$,

$$k_{AB}=\frac{\frac{2p}{k}+2pk}{\frac{2p}{k^2}-2pk^2}=\frac{k}{1-k^2}.$$ So the equation for AB is $y-\frac{2p}{k}=\frac{k}{1-k^2}\left(x-\frac{2p}{k^2}\right)$.

When $y=0$, $x=2p$. Because p is constant, so AB passes through the point C ($2p$, 0).

(b) When $k=\pm1$, A ($2p$, $2p$) and B ($2p$, $-2p$) or A ($2p$, $-2p$) and B ($2p$, $2p$) and the line AB also passes through the point $C(2p, 0)$.

Therefore, either way, AB passes through the point $C(2p, 0)$. We also know $OM\perp AB$ since $\vec{OM}\cdot\vec{AB}=0$. And when M and C do not coincide, $\angle OMC = 90°$. So the locus of all such M is the circle (minus pint O and point C) with the center at the midpoint of OC, (p, 0) and radius p. If M and c coincide, then M = ($2p$, 0). Therefore, the equation for the locus for M is $(x-p)^2+y^2=p^2$, $(x,y)\neq(0,0)$.

(2) As in Figure 14, let $\angle COM=\alpha$. Then $|OM|=2p\cos\alpha=\sqrt{3}p$. So $\cos\alpha=\frac{\sqrt{3}}{2}$. However, $\alpha\in[0,\pi)$. So, $\alpha=\frac{\pi}{6}$ and the elevated angle of AB is $\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\alpha}{3}$. Since $k_{AB}=\tan\frac{2\pi}{3}=-\sqrt{3}$, the

equation for line AB is $y = \sqrt{3}(x - 2p)$. As in Figure 15, because of symmetry, the angle of elevation of line AB is $\frac{\pi}{3}$ and the equation for line AB is $y = \sqrt{3}(x - 2p)$.

2008 Middle School
Test 2

I. Multiple-Choice Questions – There is only one right answer to each of the following 10 questions. Please write down the corresponding letter in the provided space on the left of each question.

____ 1. We can obtain the number "9" by rotating the number "6" 180° and obtain the number "6" by rotating the number "9" 180°. If we rotate the number "69" 180°, which number do we get?
(A) 69 (B) 96 (C) 66 (D) 99

____ 2. If the system of equations $\begin{cases} x + ay + 1 = 0 \\ bx - 2y + 1 = 0 \end{cases}$ has infinitely many solutions for x and y, then what must be the values for a and b?
(A) $a = 0, b = 0$ (B) $a = -2, b = 1$ (C) $a = 2, b = -1$ (D) $a = 2, b = 1$

____ 3. An isosceles triangle AOB ($OA = OB$) is situated on a coordinate system with O on the origin and A on the point (a, b). If the midline of the base AB is on the angle bisector of Quadrants I and III, then B is on which coordinate point?
(A) (b, a) (B) $(-a, -b)$ (C) $(a, -b)$ (D) $(-a, b)$

____ 4. Given two sequences of numbers: (1) 1, 3, 5, 7, …, 2007; (2) 1, 6, 11, 16, … , 2006. Then how many numbers appear in both sequences?
(A) 201 (B) 200 (C) 199 (D) 198

____ 5. AB has a fixed length. Among all the triangles with AB as its one side and this side is twice as long as another side, the triangle with the largest area will have the ratio of its three sides as one of the following.
(A) $1 : 2 : 3$ (B) $1 : 1 : 2$ (C) $1 : \sqrt{3} : 2$ (D) $1 : 2 : \sqrt{5}$

____ 6. If there are 24 bills consisting of \$10's, \$20's, and \$50's and if the amount totaled to \$1000, then how many \$20 bills are there?
(A) 2 or 4 (B) 4 (C) 4 or 8 (D) any even number between 2 to 46

____ 7. How many different 4-digit numbers can be formed by using the numbers 1, 2, 3 as its digits and each of these numbers must appear at least once?
(A) 33 (B) 36 (C) 37 (D) 39

____ 8. The rectangle $ABCD$ in Figure 1(a) has length $AD = 9$ cm. and $AB = 3$ cm.. If we fold vertex D to vertex B as in Figure 1(b), then which of the following choices represents the lengths DE and EF in that order ?

(A) 5 cm., $\sqrt{10}$ cm. (B) 5 cm., 3 cm. (C) 6 cm., $\sqrt{10}$ cm. (D) 5 cm., 4 cm.

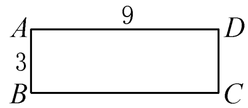


Figure 1(a)

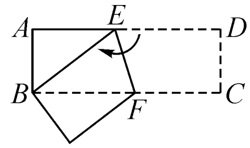


Figure 1(b)

____ 9. As in Figure 2, the graph of a function $y = mx - 4m$ intersects the x-axis at M and intersects the y-axis at N. If we drop a perpendicular line from each of two points A and B on MN to points A_1 and B_1, respectively, on the x-axis and if $OA_1 + OB_1 > 4$, then the area of ΔOA_1A, called it S_1, and the area of ΔOB_1B, called it S_2, have a ratio of one of the followings.

(A) $S_1 > S_2$

(B) $S_1 = S_2$

(C) $S_1 < S_2$

(D) Cannot be determined

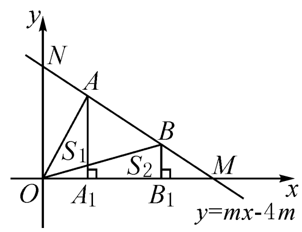


Figure 2

____ 10. If a is a real solution for $x^3 + 3x - 1 = 0$, then the straight line $y = ax + 1 - a$ will not pass one of the following regions.

(A) Quadrant I (B) Quadrant II (C) Quadrant III (D) Quadrant IV

II. Fill In the Blank Questions – Write the appropriate answer to fill in the blank space provided.

11. Simplify: $\left(\frac{7}{3}\right)^{1004}\sqrt{\frac{3^{2008}+15^{2008}}{7^{2008}+35^{2008}}} =$ ________.

12. If we get $\overline{ab8}$ by doubling a 3-digit number $\overline{3ab}$, then $\overline{3ab} =$ ________ .

13. When $x > 2$, $\sqrt{x+2\sqrt{x-1}} + \sqrt{x-2\sqrt{x-1}}$ can be simplified to ________.

14. If $f(x) = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2}$ and $f(a) = 0$, then $a =$ ______.

15. If the sum of a 4-digit natural number and 17 is a perfect square and the difference between it and 72 is also a perfect square, then this 4-digit number is _______.

16. Given an isosceles triangle ABC with BC as the base. If we fold C to a point D on AB along a straight line that passes through B and if the non-overlapped portion of ΔABC after the fold also forms an isosceles triangle, then $\angle A$ = _______°

17. One hundred ping pong balls are placed in n boxes such that the numbers of balls in each box all have the digit 8 in it. For example, if $n = 3$, these three boxes could have 8, 8 and 84 as numbers of balls. Now when $n = 5$ and two and only two boxes have the same number of balls in them, then these boxes can have _______ as their numbers of balls.

18. Given a set of ordered numbers (a, b, c, d). Use the following formulas to transform these numbers to (a_1, b_1, c_1, d_1). $a_1 = a + b$, $b_1 = b + c$, $c_1 = c + d$, $d_1 = d + a$. Follow this pattern and obtain (a_2, b_2, c_2, d_2), … , (a_n, b_n, c_n, d_n). If $1000 < \frac{a_n + b_n + c_n + d_n}{a + b + c + d} < 2000$, then n = _______.

19. As in Figure 3, OB and AB are both mirrors where $A = (1, 0)$ and $B = (0, 1)$. A ray of light is released from point O toward the mirror AB. It was reflected from point D on the surface of AB and the ray was directed towards mirror OB. After it was reflected from OB, the ray passed the point A exactly. Then the coordinates of D must be _______.

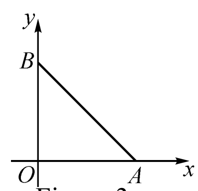


Figure 3

20. A straight highway has A_1, A_2, … , A_{11} a total of 11 bus stops. And $A_iA_{i+2} \le 12\text{km}$ (i = 1, 2, 3, … , 9) and $A_iA_{i+3} \ge 17\text{km}$ ($i = 1, 2, 3, \ldots, 8$). If $A_1A_{11} = 56\text{km}$, then $A_{10}A_{11} + A_2A_7$ = _________ km.

III. Word Problems – Show all calculation.

21. In Figure 4, ABC is a triangle, $\angle ACB = 90^\circ$, $AC = BC = 10$, CD is a ray, $\angle BCF = 60^\circ$, D is on AB, AF and BE are perpendicular to CD (or its extension) at F and E, respectively. Find the length of EF.

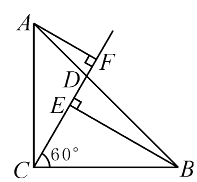


Figure 4

22. As in Figure 5, ΔABC, with A on the x-axis and C on the y-axis satisfies the following three conditions: $\angle C = 90^\circ$, $AC = 2$, $BC = 1$. If we allow A to move along the positive x-axis, then C must also move along the positive y-axis to maintain these 3 conditions.

(1) When A is at the origin, find the length of OB;

(2) When $OA = OC$, find the length of OB;

(3) When will OB be maximal? What is that maximum distance?

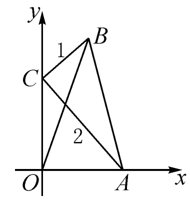


Figure 5

23. Suppose m and n are positive integers with $m > n$.

(1) If 3^m and 3^n have a common unit digit, find the smallest value for $m + n$;

(2) If 3^m and 3^n have both a common unit and ten's digits, find the smallest value for $m - n$.

2008 Middle School
Test # 2 Solutions

1. A
2. B
3. A
4. A
5. D
6. B
7. B
8. A
9. A
10. D
11. 1
12. 374
13. $2\sqrt{x-1}$
14. ± 2
15. 2008
16. 36°
17. 8, 8, 18, 28, 38
18. 10
19. $\left(\frac{1}{3}, \frac{2}{3}\right)$
20. 34
21. $EF = CF - CE = 5\left(\sqrt{3}-1\right)$
22. (1) $\sqrt{5}$ (2) $\sqrt{5}$ (3) $1+\sqrt{2}$
23. (1) 6 (2) 20

(c) Rotate the number "69" 180° and we still get the same number. (A)

) Combine the system of equations: $\begin{cases} x + ay + 1 = 0 & (1) \\ bx - 2y + 1 = 0 & (2) \end{cases}$

$(1)\times 2 + (2)\times a$ and we have $(2+ab)x = -(2+a)$. If $2+ab \neq 0$, then there is only one solution. If $2 + ab = 0$ and $2 + a \neq 0$, then there is no solution. If there are infinite number of solutions, then both $2 + ab = 0$ and $2 + a = 0$. In this case $a = -2$ and $b = 1$. (B)

(e) Because $\triangle OAB$ is an isosceles triangle with vertex O, $OA = OB$ and AB as its base, so AB is perpendicular to the line $y = x$. Reflect A (a, b) along this line will yield B (b, a). (A)

(f) From observation, the numbers that appear in both sequences are 1, 11, 21, …, 2001. Each adjacent numbers differ by 10 so the total number of such numbers is $\frac{2001-1}{10}+1=201$. (A)

(g) As in Figure 6, fix side AB. Rotate $AC = \frac{1}{2}AB$ around A. Then obviously the triangle that has the largest area has $\angle BAC$ = right angle. So, the ratio of $AC : AB = 1 : 2$ and $BC = \sqrt{5}$ and the ratio of its three sides is $1 : 2 : \sqrt{5}$. (D)

(h) Let the number of \$10, \$20, and \$50 be x, y, and $24 - (x + y)$, respectively. Then $10x + 20y + 50(24 - x - y) = 1000$ or $4x + 3y = 20$ or $x = \frac{(20 - 3y)}{4} = 5 - \frac{3y}{4}$ with $x > 0$ and $y > 0$. Since x and y are both whole numbers, y must be 4. (B)

This kind of 4-digit numbers must have exactly 2 identical digits like 1, 1, 2, and 3.
If the common digit is 1, then these four digits are 1, 1, 2, 3. If the two 1's are next to each other, then there are 6 such numbers: 1123, 1132, 2113, 3112, 2311, and 3211. If the two 1's are not adjacent, then there are also 6 such numbers: 1213, 1312, 1231, 1321, 2131, and 3121. This is a total of 12 numbers with two 1's. Similarly, there are also a total of 12 numbers for two 2's and 12 numbers with two 3's. Therefore, the total number is 36. (B)

8. As in Figure 7, let $ED = x$. Then $AE = 9 - x$ and $BE = ED = x$. Since $\triangle ABE$ is a right triangle, we have $3^2 + (9 - x)^2 = x^2$. Solving for x, $x = 5$ or $ED = 5$ cm. Let G be a point on BC so that $EG \perp BC$, $BF = ED = 5$cm., $BG = AE = 4$ cm. So, $FG = 1$cm. In right triangle EFG, $EF = \sqrt{EG^2 + GF^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$ cm. (A)

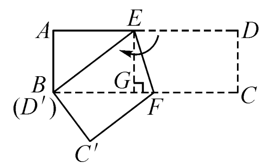


Figure 7

9. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Then $y_1 = mx_1 - 4m$ and $y_2 = mx_2 - 4m$.
The areas $S_1 = \frac{1}{2} OA_1 \cdot A_1A = \frac{1}{2} x_1(mx_1 - 4m)$ and $S_2 = \frac{1}{2} OB_1 \cdot B_1B = \frac{1}{2} x_2(mx_2 - 4m)$. So, $S_1 - S_2 = \frac{1}{2}m(x_1^2 - x_2^2) - 2m(x_1 - x_2)$. Since $m < 0$ and $x_1 < x_2$ and $x_1 + x_2 > 4$, $S_1 > S_2$. (A)

3. When $x \le 0$, $x^3 + 3x - 1 > 0$. So, there is no solution for this equation. When $x \ge \frac{1}{3}$, $x^3 + 3x - 1 > 0$. So, this equation's real roots must be inside the interval $\left(0, \frac{1}{3}\right)$. So, $0 < a < \frac{1}{3}$ and $y = ax + 1 - a$, $a > 0$ and $1 - a > 0$. Therefore, the line does not pass through the 4th Quadrant. (D)

4. $\frac{3^{2008} + 15^{2008}}{7^{2008} + 35^{2008}} = \frac{3^{2008}(1 + 5^{2008})}{7^{2008}(1 + 5^{2008})} =$ ——. So the original problem has a value of $\left(\frac{7}{3}\right)^{1004}\left(\frac{3}{7}\right)^{1004} = 1$.

5. From the original problem, $2 \times (300 + 10a + b) = 100a + 10b + 8$ or $10a + b = 74$. So the original number is 374.

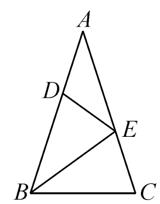
6. $\sqrt{x + 2\sqrt{x - 1}} + \sqrt{x - 2\sqrt{x - 1}} = \sqrt{(x - 1) + 2\sqrt{x - 1} + 1} + \sqrt{(x - 1) - 2\sqrt{x - 1} + 1}$

$=\sqrt{\left(\sqrt{x-1}+1\right)^2}+\sqrt{\left(\sqrt{x-1}-1\right)^2}$. $\sqrt{x-1}-1>0$ since $x>2$. Therefore, the original expression is $\left(\sqrt{x-1}+1\right)+\left(\sqrt{x-1}-1\right)=2\sqrt{x-1}$.

7. $f(x)=\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}=\frac{2-x^2}{x(x+1)(x+2)}$. $f(a)=0$ implies $2-a^2=0$ or $a=\pm\sqrt{2}$.

8. Let a be that 4-digit natural number. Then $\begin{cases} a+17=m^2 \\ a-72=n^2 \end{cases}$ where m and n are natural numbers and $m>n$. Subtract these two equations, $m^2-n^2=89$. However, 89 is a prime number so $\begin{cases} m+n=89, \\ m-n=1. \end{cases}$ which means $m=45$ and $n=44$. Therefore, $a=2008$.

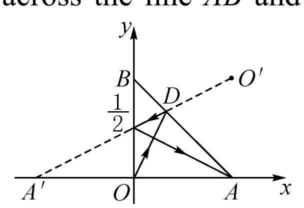
9. As in Figure 8, if C lands at D after folding and BE is the fold-line, then $\angle BDE = \angle BCE = \angle DBC$. If $\angle A$ is the vertex angle in the isosceles triangle ADE, then $DE//BC$ and $\angle ADE = \angle DBC$. However, that would make $\angle ADE = \angle BDE = 90^\circ$ which is a contradiction. If $\angle AED$ is the vertex angle, then $\angle A = \angle ADE = 180^\circ - \angle BDE = 180^\circ - \angle ACB$ which is also a contradiction. So, $\angle ADE$ must be the vertex angle. And $2\angle A + \angle ADE = 2\angle A + 180^\circ - (180^\circ - \angle A) \div 2 = 180^\circ$ or $\angle A = 36^\circ$



10. If the numbers of ping pong ball in these 5 boxes all have a digit 8, then either the unit digit of each of the 5 numbers are 8 or one of the numbers' tens digit is 8. However, the latter case is not possible so each box was put in 8 balls and then allocated the rest of 60 balls into each box and we have 8, 8, 18, 28, 38.

18. We know that $a_1+b_1+c_1+d_1=2(a+b+c+d)$. So, $\frac{a_1+b_1+c_1+d_1}{a+b+c+d}=2$. Similarly, $\frac{a_2+b_2+c_2+d_2}{a+b+c+d}=4$, $\frac{a_3+b_3+c_3+d_3}{a+b+c+d}=8$, $\cdots$, $\frac{a_n+b_n+c_n+d_n}{a+b+c+d}=2^n$. Therefore, $1000<2^n<2000$ or $n=10$.

22. As in Figure 9, $O'(1,1)$ is the image point of O (0, 0) reflecting across the line AB and $A'(-1,0)$ is the image point of A (1, 0) reflecting across the y-axis. The equation for AB is $y=-x+1$ and the equation for $A'O'$ is $y=\frac{1}{2}(x+1)$. From $\begin{cases} y=-x+1, \\ y=\frac{1}{2}(x+1). \end{cases}$ we can obtain $\begin{cases} x=\frac{1}{3}, \\ y=\frac{2}{3}. \end{cases}$ The intersection of these two lines is $D\left(\frac{1}{3},\frac{2}{3}\right)$.



23. Because $A_1A_{10}=A_1A_4+A_4A_7+A_7A_{10}$ and $A_iA_{i+3}\ge17\text{km}$, $A_1A_{10}\ge3\times17=51\text{km}$. Also, $A_1A_{11}=56\text{km}$ so $A_{10}A_{11}\le5\text{km}$. Since $A_8A_{11}\ge17\text{km}$ and $A_8A_{10}\le12\text{km}$, $A_{10}A_{11}\ge5\text{km}$. Therefore, $A_{10}A_{11}=5\text{km}$. Similarly, $A_1A_2=5\text{km}$. And $A_1A_7\ge34\text{km}$, so $A_2A_7\ge29\text{km}$. Also, $A_2A_{10}=56-5-5=46\text{km}$. And $A_7A_{10}\ge17\text{km}$, so $A_2A_7\le29\text{km}$. Therefore, $A_2A_7=29\text{km}$ and $A_{10}A_{11}+A_2A_7=34\text{km}$.

24. In the right triangles $\triangle AFC$ and $\triangle CBE$, $AC = BC$, $\angle ACF = 90^\circ - 60^\circ = 30^\circ$, and $\angle CBE = 30^\circ$ so that $\angle ACF = \angle CBE$. Therefore, $\triangle AFC \cong \triangle CBE$ which means CE = AF = 5 and BE = CF = $5\sqrt{3}$ and EF = CF – CE = $5(\sqrt{3} - 1)$.

25. (1) When A is at the origin as in Figure 10, AC is on the y-axis and $BC \perp y$-axis. So, Point $B = (1, 2)$ and $OB = \sqrt{x^2 + y^2} = \sqrt{5}$.

(2) When $OA = OC$ as in Figure 11, $\triangle OAC$ is an isosceles triangle with $AC = 2$. So, $OA = OC = \sqrt{2}$ and $\angle 1 = \angle 2 = 45^\circ$. Construct perpendicular lines from B and C to x-axis and y-axis, respectively, intersecting at D. Point $B = \left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ since $\angle 3 = 45^\circ$ and $BC = 1$ and $CD = BD = \frac{\sqrt{2}}{2}$. Therefore, $OB = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{3\sqrt{2}}{2}\right)^2} = \sqrt{5}$.

(3) As in Figure 12, Take the midpoint E of AC and connect OE and BE. In Right Triangle $\triangle AOC$, OE is the median to the hypotenuse AC. So, $OE = \frac{1}{2}AC = 1$. In $\triangle ABC$, $BC = 1$, $CE = \frac{1}{2}AC = 1$, and $\angle BCE = 90^\circ$. Therefore, $BE = \sqrt{2}$. If O, E, and B are not collinear, then $OB < OE + EB = 1 + \sqrt{2}$. If O, E, and B are collinear, then $OB = OE + EB = 1 + \sqrt{2}$. Therefore, OB takes the maximum value of $1 + \sqrt{2}$ when O, E, and B are collinear.

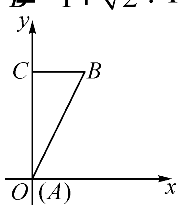


Figure 10

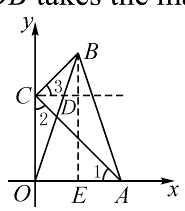


Figure 11

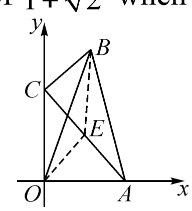


Figure 12

26. (1) Since the last digit of 3^m and 3^n are the same, we know $3^m - 3^n$ is a multiple of 10. That means $3^m - 3^n = 3^n(3^{m-n} - 1)$ is a multiple of 10. However, 3^n and 10 are relatively primes, $3^{m-n} - 1$ must be a multiple of 10. That means, the unit digit of 3^{m-n} must be 1. Since $3^4 = 81$ satisfies this condition, the smallest value of $(m - n)$ must be 4. Therefore, $m + n$ takes on the smallest value of 6 when $n = 1$ and $m = 5$.

(2) Following the logic from (1), $3^m - 3^n = 3^n(3^{m-n} - 1)$ is a multiple of 100. Since 3^n and 100 are also relatively primes, $3^{m-n} - 1$ must be a multiple of 100. That means, the last two digits of 3^{m-n} must be 01. Because the last digit of 3^{m-n} is 1, so $(m - n)$ must be a multiple of 4. Let $(m - n) = 4t$ where t is a positive integer. Then $3^{m-n} = 3^{4t} = 81^t$ with its last two digits is 01. However, when $t = 1, 2, 3, 4$, the last two digits of 81^t are not 01. $t = 5$ is the smallest positive integer so that 81^t has 01 as its last two digits. Therefore, $t = 5$ and $(m - n) = 20$.

2008 Primary School
Test 2

I. Fill In the Blank Questions – Write the appropriate answer to fill in the blank space provided.

1. $\left(1\frac{3}{2008}+2\frac{3}{1004}+8\frac{3}{251}\right)\div\left(1\frac{1}{2008}+2\frac{1}{1004}+8\frac{1}{251}\right)=$ ______.

2. There are _______ different ways one can place 5 different objects from left to right.

3. Given a sequence of numbers: 1, 1, 3, 8, 22, 60, 164, 448, … . Starting with the 4th number, each number is twice the sum of the two numbers that precede it. The 10th number in this sequence is ______.

4. A row of chairs has 27 seats. One has to sit at least ______ people in these seats so that any one who seated later will be sure to have someone sitting in the next seat.

5. A tightly closed bottle is holding some water inside as in Figure 1. From the numbers given in the figure, one can deduce that the total volume of the bottle is ______ cm^3. (Use $\pi = 3.14$)

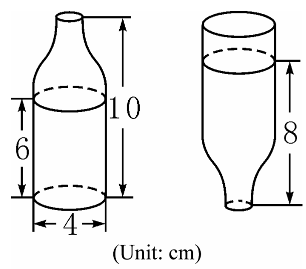


Figure 1

6. Figure 2 shows a field in the shape of a trapezoid with the indicated dimensions. The area of this field is _______ m^2.

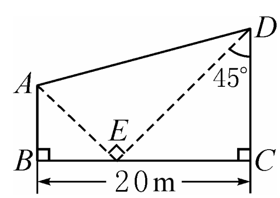


Figure 2

7. Figure 3 shows four cubes stacked together. Each cube has lengths of 1 cm, 2 cm, 3 cm, and 5 cm, respectively. Then the surface area of this solid is _______ cm^2.

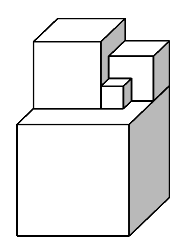


Figure 3

8. Fifth grade has a total of 36 students and 5 special interest groups. The groups are A, B, C, D, and E. Each student in class participates in one and only one of these 5 groups. Group A is the most popular group with 15 students. B is the second most popular group. Groups C and D have the same number of students in them. E is the least popular of the 5 groups with only 4 students. Then group B has ________ students.

9. People are picking all the apples from a field. After they had picked $\frac{2}{5}$ (in weight) of all the apples, these apples filled 3 cases with 16 kilograms left over. The rest of the apples filled another 6 cases. The total weight of all the apples is _______ kilograms.

10. A company is constructing a highway. The original plan is to construct 720 meters a day. However, the actual work has gone better than expected and the company was able to construct 80 meters more per day than originally planned. So they had finished the work 3 days earlier. This highway is ______ kilometers long.

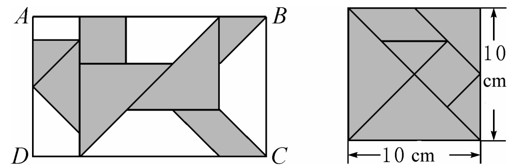
11. Mr. Wong is driving from Beijing to Shanghai. In the beginning, he drives the car $\frac{1}{9}$ faster than what he had originally planned. So he arrives in Shanghai one and a half hours earlier. When he returns from Shanghai to Beijing, he drives the same speed as his original plan for 280 kilometers and then he increases his speed by $\frac{1}{6}$. He arrives in Beijing 1 hour and 40 minutes earlier than planned. The distance between Beijing and Shanghai is _______ kilometers.

12. Two identical rectangular boxes with 5 cm. in length, 4 cm. in width, and 3 cm. in height. If we glue these two boxes together to form another rectangular box, among all the possible boxes, the one with the least amount of surface area has _______ cm^2.

II. Word Problems – Show all calculation.

13. The famous Goldbach Hypothesis states that, “Any even number greater than or equal to 4 can be written as the sum of two prime numbers.” For example, 6 = 3 + 3, 12 = 5 + 7, etc. Then, how many ways can 100 be written as the sum of two distinct prime numbers? Write out all these possible sums. (100 = 3 + 97 and 100 = 97 + 3 are considered as one way.)

14. Figure 4(a) shows a rectangle. The shaded part is put together by a set of tangram (see Figure 4(b)) with area of 100 cm^2. What is the area of rectangle *ABCD*?



(a) (b)

Figure 4

15. Four table tennis players with uniform numbers of 2005, 2006, 2007 and 2008. During the competition, the rules require two players to play each other in a number of games that is equal to the remainder when the sum of their uniform numbers is divided by 4. In that case, how many games did 2008 player play?

16. Nine identical pipes are placed in a tank. One is for water to enter the tank (called in-pipe) and the other 8 (called out-pipes) are to drain water from the tank. In the beginning, the in-pipe is turned on and water is entering into the tank at a constant rate. Then, to drain all the water from the tank, out-pipes are turned on. If all 8 out-pipes are turned on, then it would take 3 hours to drain the tank. If 5 out-pipes are turned on, then it would take 6 hours. If we want to drain the tank in 4.5 hours, how many out-pipes must be turned on? (The in-pipe is on during the whole process.)

2008 Primary School Test # 2 Solutions

1. 1 2/2009
2. 120
3. 3344
4. 9
5. 100.48
6. 200
7. 194
8. 7
9. 160
10. 21.6
11. 1260
12. 148
13. $100 = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53$
14. 187.5 cm^2
15. 6
16. 6