SUMS OF POWERS AND BERNOULLI NUMBERS

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1. Fermat and Pascal

On September 22, 1636 Fermat claimed in a letter that he could find the area under any "higher" parabola and Roberval wrote back to say that he had discovered the same thing by using the inequality $\frac{n^{k+1}}{k+1} < 1^k + 2^k + 3^k + \cdots + n^k < \frac{(n+1)^{k+1}}{k+1}$. Not to be outdone, Fermat responded with a set of equations saying "All these propositions, however pretty in themselves, have aided me in the quadrature that I am pleased you value". See Edwards [**?**]. Fermat is referring here to actually finding a closed form for the sum of the powers. Although he did not provide a proof he did indicate that he used the patterns that arise in the figurate numbers,i.e., the triangular numbers, the pyramidal or tetrahedral numbers which are sums of triangular numbers, and so on. So before Newton and Leibniz have discovered the Fundamental Theorem of Calculus, Fermat had already solved the tangent line problem and area problem for polynomials.

Thus the table of numbers which we call Pascal's Triangle was well-known long before Pascal wrote it down in his treatise on the "arithmetical triangle" in 1654. The name "Arithmetical Triangle" had been given to the configuration in 1630 by de Moivre and that in turn gave rise to the title of Pascal's book. However the table itself is much older and can be found in India, China, Arabia and Japan hundreds of years previous to this. Nevertheless the insights and connections to binomial coefficients, combinatorics and probability that Pascal revealed in this treatise more than justify renaming it *Pascal's Triangle*. He presented 12 theorems with proof including the binomial theorem for positive exponents and a combinatorial proof of the recursive formula for forming the triangle. He also showed how to apply the triangle by using it to answer an outstanding problem on how to divide a bet when a game of chance had been stopped before a winner was determined. In a further treatise he shows how to use the binomial coefficients to find the sum of the k^{th} powers of the first $n-1$ positive integers if the formulas for the sums of the powers less than k are known. For example, knowing the sum of the first n natural numbers is $\frac{n(n+1)}{2}$ and $(n+1)^3 - n^3 = 3n^2 + 3n + 1$, to calculate the sum of the squares Pascal proceeded as follows.

$$\begin{array}{ccccccccc}
n^3 & - & (n-1)^3 & = & 3(n-1)^2 & + & 3(n-1) & + & 1 \\
(n-1)^3 & - & (n-2)^3 & = & 3(n-2)^2 & + & 3(n-2) & + & 1 \\
 & \vdots & & & & \vdots & & & \\
2^3 & - & 1^3 & = & 3(1)^2 & + & 3(1) & + & 1 \\
\hline
n^3 & - & 1^3 & = & 3\displaystyle\sum_{k=1}^{n-1} k^2 & + & 3\displaystyle\sum_{k=1}^{n-1} k & + & n-1
\end{array}$$

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This can be solved for the sum of the squares and simplified.

$$\sum_{k=1}^{n-1} k^2 = \frac{1}{3}\left(n^3 - n - 3\sum_{k=1}^{n-1} k\right) = \frac{1}{3}n^3 - \frac{1}{3}n - \frac{n(n-1)}{2} = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$$

This procedure of Pascal's can be generalized and became the standard method used in algebra books after he presented it in a separate treatise *Potestatum Numericarum Summa*, "Sums of Powers of Numbers". It is interesting to note how telescoping is used to remove all but the first and last term on the left. If you look at the usual proof of the Fundamental Theorem of Calculus you will see essentially the same technique. I first encountered this technique in one of Polya's famous books on problem solving. After criticizing the solution as a "device" coming from "out of the blue", he goes on to show how one might come up with such an idea. He finishes with "A method is a device which you use twice."

Exercise 1. If you have studied upper and lower Riemann sums, use Roberval's inequality to evaluate $\int_0^1 x^n\, dx$.

Exercise 2. The triangular numbers, t_n are found by summing the first n natural numbers: $\{1, 3, 6, \ldots, \frac{n(n+1)}{1\cdot 2}, \ldots\}$. The pyramidal numbers are given by $p_n = t_1 + t_2 + t_3 + \cdots + t_n$. Show that $p_n = \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}$. Generalize this and find these numbers in Pascal's Triangle.

Exercise 3. From Pascal's Triangle we have

$$(n+1)^{k+1} - (n)^{k+1} = \binom{k+1}{1}n^k + \binom{k+1}{2}n^{k-1} + \cdots + \binom{k+1}{k}n + \binom{k+1}{k+1}.$$

Show by successively substituting $n-1, n-2, \ldots, 2, 1$ in for n and adding that Pascal's method can be generalized to

$$(k+1)\sum_{j=1}^{n-1} j^k = n^{k+1} - n - \binom{k+1}{2}\sum_{j=1}^{n-1} j^{k-1} - \cdots - \binom{k+1}{k}\sum_{j=1}^{n-1} j.$$

Exercise 4. Substitute $n+1$ for n in the equation above in Exercise **??** and use the new equation to prove the right-hand side of Roberval's inequality.

Exercise 5. Prove Bernoulli's inequality which states that for $x \geq -1$ and $k > 0$,

$$(1+x)^k \geq 1 + kx.$$

This is clearly true by the binomial theorem for $x \geq 0$. Use induction for $-1 \leq x < 0$.

2. Faulhaber and Bernoulli

By 1631, unbeknownst to Pascal and Fermat the "Arithmetician of Ulm", Faulhaber had already found the formulas for the sum of the powers up to 17. Jakob Bernoulli knew of Faulhaber's work and included, with credit, the formulas in his *Ars Conjectandi*, "The Art of Conjecturing" which was published posthumously in 1713. After showing how to derive the formulas in a manner similar to Pascal, he writes out the formulas for the powers up to 10. (The $\int$ stands for sum and $*$ indicates a missing power. Note that he is adding the powers of n numbers instead of $n-1$ numbers) At this point he says "Indeed, a pattern can be seen in the progressions herein, which can be continued by means of this rule. . . .' See *Mathematical Masterpieces* [**?**]. Can you see what Bernoulli saw? Look at the formulas on the next page. Certain patterns are clear. What is the degree of the polynomial? What are the coefficients of the highest degree term and the next to highest degree term? Which terms are missing? Now it becomes confusing. What are the coefficients in the third and following columns? How can the last coefficient in a row be found?

$$\int n = \frac{1}{2}nn + \frac{1}{2}n.$$
$$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n.$$
$$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn.$$
$$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 * -\frac{1}{30}n.$$
$$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 * -\frac{1}{12}nn.$$
$$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 * -\frac{1}{6}n^3 * +\frac{1}{42}n.$$
$$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 * -\frac{7}{24}n^4 * +\frac{1}{12}nn.$$
$$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 * -\frac{7}{15}n^5 * +\frac{2}{9}n^3 * -\frac{1}{30}n.$$
$$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 * -\frac{7}{10}n^6 * +\frac{1}{2}n^4 * -\frac{3}{20}nn.$$
$$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 * -1n^7 * +1n^5 * -\frac{1}{2}n^3 * +\frac{5}{66}n.$$

Here is what Bernoulli saw.

$$\sum_{i=1}^{n} i^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \sum_{j=2}^{n} \frac{1}{k+1}\binom{k+1}{j} B_j n^{k+1-j} \qquad \text{for } k \geq 1,$$

where $B_k \in \{\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0 \dots\}$. By setting $n = 1$ the left-hand side becomes 1 and the last coefficient, B_k, can be found from the proceeding ones.

Exercise 6. Complete the next two rows in Bernoulli's table. Note that the $\frac{5}{66}$ becomes $\frac{10}{132}$ which makes the seventh column continue $\frac{55}{132}$, $\frac{220}{132}$,.... You should then be able to find that $B_{12} = -\frac{691}{2730}$.

Exercise 7. Bernoulli claimed to have found the sum of the tenth powers of the first thousand integers in less than 7.5 minutes. Let $n = 1000$ in the final row of Bernoulli's table and see if you can match Bernoulli. Do this with pencil and paper only! (If you accept this mission you should get 91,409,924,241,424,243,424,241,924,242,500.)

Exercise 8. Roberval's inequality can be proved completely using Bernoulli's inequality to conclude $(1 + \frac{1}{n})^{k+1} > 1 + \frac{k+1}{n}$ and $(1 - \frac{1}{n})^{k+1} > 1 - \frac{k+1}{n}$. (Hint: Multiply the each inequality by n^{k+1} and solve for n^k. Combine the results to get the compound inequality

$$\frac{n^{k+1} - (n-1)^{k+1}}{k+1} < n^k < \frac{(n+1)^{k+1} - n^{k+1}}{k+1}$$

Now substitute $1, 2, 3, \dots, n$ into the inequalities and add the n inequalities.)

3. Euler

One of the series problems Jakob Bernoulli was unable to solve was to find the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$. In his treatise of 1689 he did show that the sum was between 1 and 2 and said he would be most gratified if someone would solve it. In 1731, Euler using logarithms, integration, and power series was able to show $\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}k^2} + (\log 2)^2$. See Dunham [**?**]. With this fast converging series he was able compute the sum to 6 decimal places (over 10^6 terms of the original series would have to be added to get this kind of accuracy). At this time Euler is on his way to discovering the *Euler-Maclaurin Summation Formula.* Later he will recognize the connection with Bernoulli's work and will name the numbers we have seen the *Bernoulli Numbers.* Using this technique he is able to calculate the sum to 20 decimal places and compare it with the amazing fact he has discovered in 1735 that "six times the sum is the square of π". With this result Euler was elevated to the position of number one in the world of mathematics, but he did not rest there. This was in some ways just the beginning. Euler continued to produce world class mathematics for the next 48 years. Among the many things he proved were that when $\frac{x}{e^x-1}$ is expanded as a power series the result is $\sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$ for $|x| < 2\pi$. The result for the sum of the reciprocals of the squares was generalized to show $\sum_{k=1}^{\infty} = \frac{1}{k^{2n}} = \frac{4^n |B_{2n}| \pi^{2n}}{2(2n)!}$ for $n \in \mathbb{N}$

Exercise 9. Use $(\log 2)^2 + \sum_{k=1}^{14} \frac{1}{2^{k-1}k^2}$ to approximate $\sum_{k=1}^{\infty} \frac{1}{k^2}$ to 6 decimal places.

Euler shows in the second volume of his famous textbook *Institutiones Calculi Differentialis.* See a translation of this chapter in [**?**]. what he did to estimate $s = \sum_{k=1}^{\infty} = \frac{1}{k^2}$. Let $s = \int \frac{1}{x^2}\, dx + B_1 \frac{1}{x^2} - B_2 \frac{1}{x^3} + B_3 \frac{1}{x^5} - \cdots$ where $x^{-(n+2)}$ comes from the higher derivatives of x^{-2}, i.e., $\frac{1}{(n+1)!} \frac{d^n x^{-2}}{dx^n}$. Euler notes that the B_k's become infinitely large, but he intends to only add the terms "until they begin to diverge". It was his genius that could detect when that occurred. Later study of divergent series vindicates his method.

Exercise 10. To do the same calculation Euler did, solve for C the constant of integration, let $x = 10$ and s be the actual sum of the first 10 terms to 20 decimal places (1.5497677311665406904). Beginning with s add the following: $\frac{1}{x} - \frac{1}{2x^2} + \frac{B_2}{x^3} + \frac{B_4}{x^5} + \cdots + \frac{B_{16}}{x^{17}}$. If you maintain 20 decimal places you will have found the sum to 17 decimal places.

4. Bernoulli Mnemonic and Fermat Again

A simple mnemonic for the power sum formula is

$$1^k + 2^k + \cdots + n^k = \frac{(n+B)^{k+1} - B^{k+1}}{k+1}$$

where the symbol B^k, after the binomial is expanded, is replaced with B_k and $B_1 = +\frac{1}{2}$ instead of $-\frac{1}{2}$ since the sum is taken to n^k rather than $(n-1)^k$. See Conway and Guy [**?**].

Finally, we return to Fermat. A major breakthrough was made in 1850 by Kummer when he proved that whenever a prime $p > 2$ is *regular*, Fermat's last theorem, $x^p + y^p = z^p$ never has non-zero solutions, is true. A prime p is regular iff p does not divide any of the numerators of the Bernoulli numbers $B_2, B_4, \ldots, B_{p-3}$. The only irregular primes less than 100 are 37, 59 and 67. Kummer actually went on and eliminated these three cases.

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Sums of Powers Warm-ups

To warm up for this session find the following sums. It might be helpful to know how Pascal's triangle is formed. See the other side of this sheet for the first twelve rows.

1. $\sum_{k=1}^{100} k$

2. $\sum_{k=1}^{100} k(k-1)$

3. $\sum_{k=1}^{100} k(k-1)(k-2)$

4. $\sum_{k=1}^{100} k^2$

5. $\sum_{k=1}^{100} k^3$

6. $\sum_{k=2}^{100} \frac{1}{k(k-1)}$

7. $\sum_{k=3}^{100} \frac{1}{k(k-1)(k-2)}$

8. $\sum_{k=1}^{\infty} \frac{1}{k}$

9. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

10. $\sum_{k=1}^{\infty} \frac{1}{k^4}$

Sums of Powers

Fill in the next two rows of the triangle.

1											
1	1										
1	2	1									
1	3	3	1								
1	4	6	4	1							
1	5	10	10	5	1						
1	6	15	20	15	6	1					
1	7	21	35	35	21	7	1				
1	8	28	56	70	56	28	8	1			
1	9	36	84	126	126	84	36	9	1		
1	10	45	120	210	252	210	120	45	10	1	
1	11	55	165	330	462	462	330	165	55	11	1

$S_k(n) = 0^k + 1^k + 2^k + \cdots\ (n-1)^k$. Use the pattern to fill in the row for $S_6(n)$ so that the polynomial will sum the first $n-1$ positive integers to the sixth power

$$\begin{aligned}
S_0(n) &= 1n \\
S_1(n) &= \tfrac{1}{2}n^2 - \tfrac{1}{2}n \\
S_2(n) &= \tfrac{1}{3}n^3 - \tfrac{1}{2}n^2 + \tfrac{1}{6}n \\
S_3(n) &= \tfrac{1}{4}n^4 - \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 \\
S_4(n) &= \tfrac{1}{5}n^5 - \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n \\
S_5(n) &= \tfrac{1}{6}n^6 - \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2 \\
S_6(n) &= \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
S_{10}(n) &= \tfrac{1}{11}n^{11} - \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 - \ n^7 + \ n^5 - \tfrac{1}{2}n^3 + \tfrac{5}{66}n
\end{aligned}$$

To sum the powers from 1 to n, the second coefficient becomes $+\frac{1}{2}$ rather than $-\frac{1}{2}$. Use $S_{10}(n)$ to compute the sum of the first 1000 positive integers to the 10th power.