Berkeley Math Circle
Complex Generating Functions – Ivan Matić

1. Does there exist an integer n such that 3^n has 2009 consecutive zeroes in its decimal representation?

2. If a is a natural number, show that the number of positive integral solutions of the equation

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = a \tag{1}$$

is equal to the number of non-negative integral solutions of

$$y_1 + 2y_2 + 3y_3 + \cdots + ny_n = a - \frac{n(n+1)}{2} \tag{2}$$

[By a *solution* of equation (1), we mean a set of numbers $\{x_1, x_2, \dots, x_n\}$ which satisfies equation (1)].

3. Let S_n be the number of triples (a,b,c) of non-negative integers such that $a+2b+3c=n$. Calculate $\sum_{n=0}^{\infty} \frac{S_n}{3^n}$.

4. Let n and k be positive integers with $k \geq n$ and $k-n$ an even number. Let $2n$ lamps labelled 1, 2, $\dots$, $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequence of *steps*: at each step one of the lamps is switched (from on to off or from off to on).

   Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

   Let M be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

   Determine the ratio N/M.

5. If p is a prime number and $a_0, a_1, \dots, a_{p-1}$ are rational numbers satisfying

$$a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_{p-1}\varepsilon^{p-1} = 0,$$

where

$$\varepsilon = e^{\frac{2\pi i}{p}} = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p},$$

then $a_0 = a_1 = \cdots = a_{p-1}$.

6. Consider a rectangle that can be tiled by a finite combination of $1 \times m$ and $n \times 1$ rectangles, where m, n are positive integers. Prove that it is possible to tile this rectangle using only $1 \times m$ rectangles or only $n \times 1$ rectangles.

7. How many n-digit numbers, all of whose digits are 1, 2, 4, 5, have the digit sum a multiple of 5?

8. Let m and n be integers greater than 1 and let $a_1, a_2, \dots, a_n$ be integers, none of which is divisible by m^{n-1}. Prove that we can find integers $e_1, e_2, \dots, e_n$, not all zero, such that $|e_i| < m$ for all i and $m^n \mid e_1a_1 + \cdots + e_na_n$.

9. Let $p > 2$ be a prime number and let $A = \{1, 2, \dots, 2p\}$. Find the number of subsets of A each having exactly p elements and whose sum is divisible by p.

10. All elements of the matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ are integers. Assume that for each sequence of real numbers r_1, r_2, $\dots$, $r_n \in \mathbb{R}$ such that $r_1^2 + \cdots + r_n^2 \neq 0$ there exists $i \in \{1, 2, \dots, m\}$ such that $a_{i1}r_1 + a_{i2}r_2 + \cdots + a_{in}r_n > 0$. Prove that there exists a sequence $k_1, k_2, \dots, k_m$ of non-negative integers such that the following two conditions hold:

    (i) At least one of $k_1, k_2, \dots, k_m$ is bigger than 0, and

    (ii) For all $j \in \{1, 2, \dots, n\}$ the following equality holds:

$$k_1a_{1j} + k_2a_{2j} + \cdots + k_ma_{mj} = 0.$$