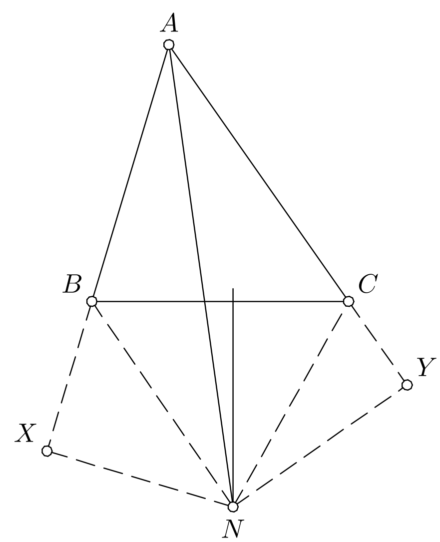
Berkeley Math Circle
Geometry of Circles – Ivan Matić

Theorem. Let M and N be the midpoints of the sides AB and AC of the triangle ABC. Then $MN \| BC$ and $MN = BC/2$.

Exercise 1. Let $ABCD$ be a quadrilateral and let M and N be the midpoints of the segments AB and CD respectively. Prove that $\overrightarrow{BC} + \overrightarrow{AD} = 2\overrightarrow{MN}$.



Exercise 2. Consider the picture on the left. $\triangle ABC$ is an arbitrary triangle and N is the intersection point of the bisector of the angle BAC and biserctor of the segment BC. Denote by X and Y feet of perpendiculars from N to AB and AC. We have that $NX = NY$ because each point on the bisector of angle $\angle BAC$ is on equal distance from the rays of those angles. Also, $NB = NC$ since each point on the bisector of the segment BC is on equal distance from the endpoints of the segment. Since $\angle NXB = \angle NYC$ we conclude that $\triangle BXN \cong \triangle CYN$ implying that $XB = YC$. Similarly, (using that $\angle XAN = \angle YAN$) we get that $\triangle AXN \cong \triangle AYN$ implying that $AX = AY$, hence $AB = AX - BX = AY - YC = AC$ and we concluded that $AB = AC$. Similarly, we prove that $AB = BC$ and the conclusion is that *every triangle is equilateral*. Hence every angle is equal to the angle of 60°, particularly $61^\circ = 60^\circ$. Subtracting number 60° from both sides we get that $1^\circ = 0^\circ$, or $1 = 0$.

Exercise 3. Let $ABCD$ be a rectangle and E the foot of perpendicular from B to AC. If F and G are midpoints of CD and AE, respectively, prove that $\measuredangle BGF = 90^\circ$.

Exercise 4. Let ABC be a triangle, and let P be a point inside it such that $\measuredangle PAC = \measuredangle PBC$. The perpendiculars from P to BC and CA meet these lines at L and M, respectively, and D is the midpoint of AB. Prove that $DL = DM$.

Exercise 5. The circle with center O touches the lines AB and AC at points B and P. Let H be the foot of perpendicular from O to BC and let T be the intersection point of OH and BP. Prove that AT bisects the segment BC.

Exercise 6. Let A_1, B_1 and C_1 be the points of tangency of the inscribed circle with the sides BC, CA and AB of the triangle ABC. If the angles of $\triangle ABC$ are α, β and γ, calculate the angles of $\triangle A_1B_1C_1$.

Exercise 7. Given an isosceles rectangular triangle ABC ($\measuredangle BAC = 90^\circ$), let E and F be two points on AB and AC such that $AE = AF$. Perpendicular from E to BF meets BC at P and perpendicular from A to BF meets BC at Q. Prove that $PQ = QC$.
Hint. Denote by K and L the feet of perpendiculars from E and C to BF. Look very carefully at the triangles AKB and ALC. After you conclude something amazing, give some attention to the triangle AKL and to the quadrilateral $PCLK$.

Exercise 8. A circle with center O passes through points A and C and intersects the sides AB and BC of the triangle ABC at points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at two distinct points B and M. Prove that $\measuredangle OMB = 90^\circ$.