Berkeley Math Circle Graph Theory and Ramsey Theory

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Definitions

1 A *graph* is a pair (V,E), where V is a finite set and E is a set of unordered pairs of elements of V. The elements of V and E are called *vertices* and *edges*, respectively. A graph is usually understood to be *simple*, having no multiple edges and no loops. Otherwise, it is called a *pseudograph*.

2 A *directed graph (digraph)* is the same as a graph, except that the edges are now ordered pairs of distinct vertices. (An edge is said to "come out of" the first vertex in the pair and "go into" the second vertex.) When we say "graph", we mean an undirected graph unless otherwise specified.

3 Two vertices are *adjacent* if they are the endpoints of an edge. The *degree* of a vertex is the number of edges it is an endpoint of. In a directed graph, the *in-degree* and *out-degree* of a vertex are the number of edges coming in and going out of, respectively, that vertex. A graph is *k-regular* if every vertex has degree k.

4 A *walk* is a sequence vertex, edge, vertex, ..., which ends with a vertex, and where the edge between any two vertices in the sequence is an edge which actually joins those two vertices. In other words, a walk is just what you think it is. The *length* of a walk is the number of edges in the walk. If the starting vertex is the same as the ending vertex, the walk is *closed*.

5 A walk with no repeated edges is called a *trail*. A walk with no repeated vertices is called a *path*.

6 A closed trail is called a *circuit*. A "closed path" is a contradiction in terms, but what this term evokes is called a *cycle*. More precisely, a cycle is a closed walk in which no vertex is repeated except for the starting vertex (which is the same as the end vertex). A cycle of length n is called an *n-cycle*.

7 A graph is *connected* if for any two vertices, there exists a walk starting at one of the vertices and ending at the other. Otherwise the graph is called *disconnected*.

8 A connected graph with no cycles is called a *tree*. If the graph is disconnected, and each connected component is a tree, then the entire graph is called a *forest*.

9 If the vertices of a graph can be partitioned into two (non-empty) subsets such that all edges of the graph connect only vertices from different sets (never two vertices from the same sets), then the graph is called *bipartite*.

10 The *complete* graph K_n is the graph on n vertices in which every pair of vertices is an edge. The *complete bipartite* graph $K_{m,n}$ is the graph on $m+n$ vertices in which every pair of vertices, one from the first m and one from the other n, is an edge.

11 A *planar* graph is one that can be drawn in the plane, with points representing the vertices, and (polygonal) curves representing the edges, so that no two edges meet except at a common endpoint. The regions into which the edges divide the plane are called *faces*.

12 An *Eulerian trail/circuit* is a trail/circuit which visits every edge of a graph. Such a graph is called Eulerian.

13 A *Hamiltonian path/cycle* is a path/cycle which visits each vertex of the graph. Such a graph is called Hamiltonian.

Several Basic Theorems

1 *Handshake Lemmma*. The sum of the degrees of the vertices equals twice the number of edges; as a corollary, if v is odd, one of the vertices has even degree.

2 For connected graphs, $e \geq v-1$, with equality holding for trees. For a forest with k connected components, $e = v-k$.

3 If $e \geq v$, then the graph has a cycle.

4 A graph is bipartite if and only if it has no odd cycles.

5 (a) A graph has an Eulerian trail if and only if it has either zero or two vertices with odd degree.

(b) A graph has an Eulerian circuit if and only if all vertices have even degree.

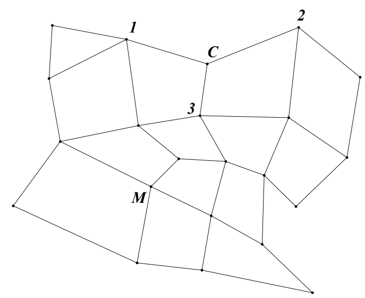
6 For a connected planar graph, $v-e+r=2$, where r denotes the number of regions (including the unbounded region) that the graph divides the plane into.

Three Games that use Graphs

For games 1 and 3, two players alternate turns. The rule for a legal move is described. The game ends when no legal moves can be made. The winner is the last player to make a legal move. Your job is to analyze the game and figure out a winning strategy.

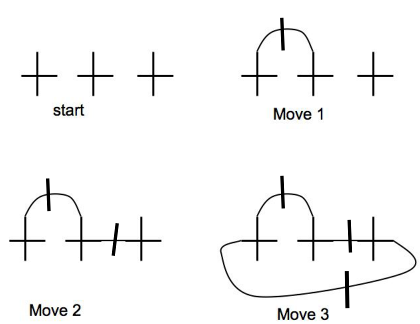
1 *Color the Grids*. You start with an $n \times m$ grid of graph paper. Players take turns coloring red one previously uncolored unit edge of the grid (including the boundary). A move is legal as long as no closed path has been created.

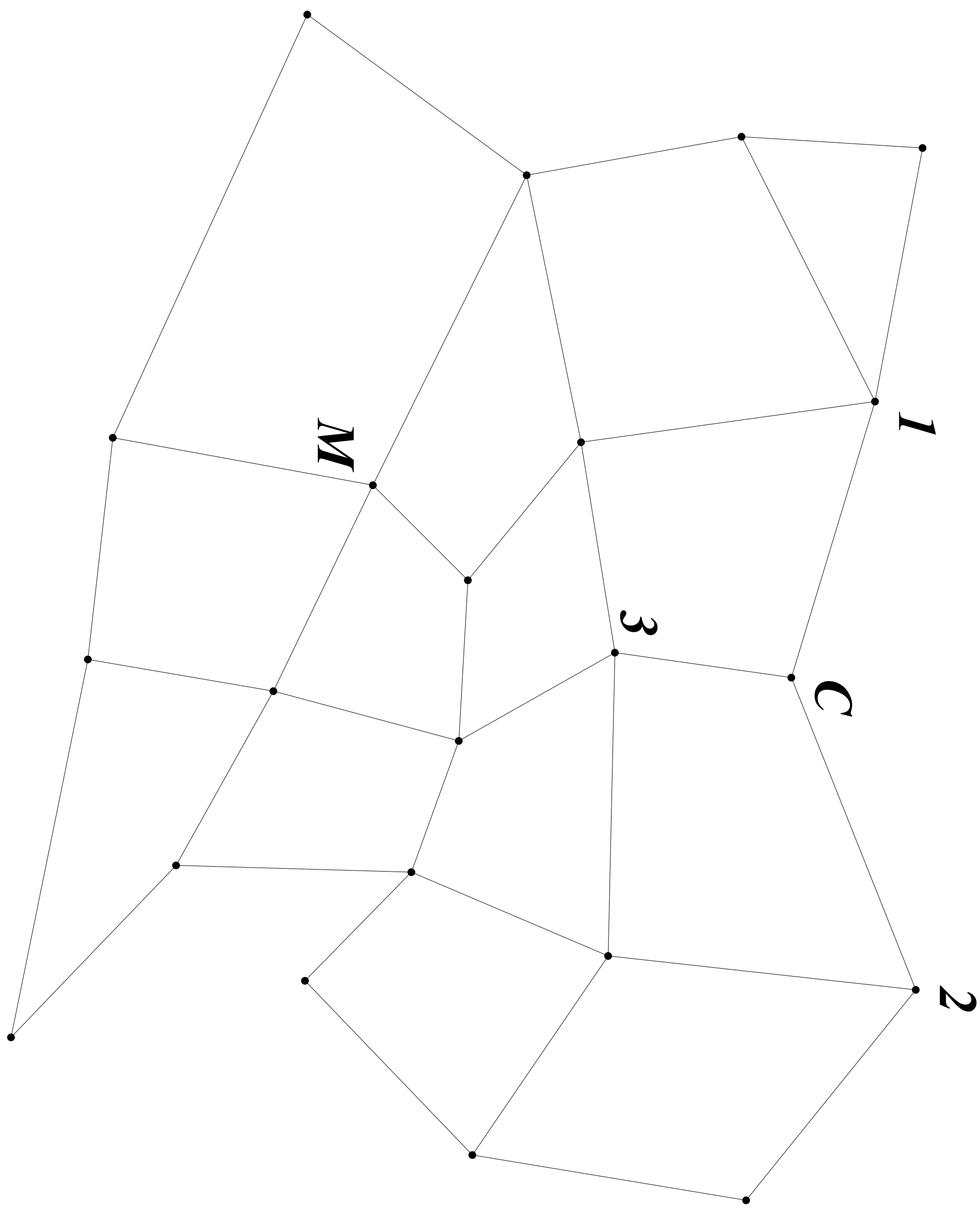
2 *Cat and Mouse*. A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below.



Initially, the cat is at the point labeled C; the mouse is at M. The cat goes first, and can move to any neighboring point connected to it by a single edge. Thus the cat can go to points 1, 2, or 3, but no others, on its first turn. The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win? (Adapted from Ravi Vakil's *A Mathematical Mosaic*. A larger diagram is on the next page, so you can play this game.)

3 *Brussels Sprouts*. Start by putting a few crosses on a piece of paper. On each move, a player can connect the two endpoints of a cross together, with a single line (which can be curved). Then a new cross is drawn on this connection line. You cannot ever draw a line that intersects another already-drawn line. Here is an example of the first few moves of a 3-cross game.




M
1
3
C
2

More Problems

4 Show that every graph contains two vertices of equal degree.

5 In the nation of Klopstockia, each province shares a border with exactly three other provinces. Can Klopstockia have 17 provinces?

6 Draw a graph with eight vertices, four of which have degree 4 and four of which have degree 3.

7 Show that it is possible to have a 4-regular graph with n vertices, for every $n \geq 5$.

8 (Colorado Springs Mathematical Olympiad) If 127 people play in a singles tennis tournament, prove that at the end of the tournament, the number of people who have played an odd number of games is even.

9 How many edges must a graph with n vertices have in order to guarantee that it is connected?

10 A large house contains a television set in each room that has an odd number of doors. There is only one entrance to this house. Show that it is always possible to enter this house and get to a room with a television set.

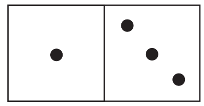
11 Show that if a graph has v vertices, each of degree at least $v/2$, then this graph is connected. In fact, show that it is Hamiltonian.

12 A *tournament* is a directed graph in which every pair of vertices occurs as an edge in one order or the other (but not both). Prove that every tournament has a (directed) Hamiltonian path. Also, which tournaments contain a Hamiltonian cycle?

13 (USAMO 1986) During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that, at some moment, some three of them were sleeping simultaneously.

Even More Problems

16 A domino consists of two squares, each of which is marked with $0, 1, 2, 3, 4, 5$, or 6 dots. Here is one example.



Verify that there are 28 different dominos. Is it possible to arrange them all in a circle so that the adjacent halves of neighboring dominos show the same number?

17 Is it possible for a knight to travel around a standard 8×8 chessboard, starting and ending at the same square, while making every single possible move that a knight can make on the chessboard, *exactly once*? We consider a move to be completed if it occurs in either direction.

18 An *n-cube* is defined intuitively to be the graph you get if you try to build an n-dimensional cube out of wire. More rigorously, it is a graph with 2^n vertices labeled by the n-digit binary numbers, with two vertices joined by an edge if the binary digits differ by exactly one digit. Show that for every $n \geq 1$, the n-cube has a Hamiltonian cycle.

19 If you place the digits 0,1,1,0 clockwise on a circle, it is possible to read any two-digit binary number from 00 to 11 by starting at a certain digit and then reading clockwise. Is it possible to do this in general?

20 *BAMO 2004*. NASA has proposed populating Mars with 2,004 settlements. The only way to get from one settlement to another will be by a connecting tunnel. A bored bureaucrat draws on a map of Mars, randomly placing N tunnels connecting the settlements in such a way that no two settlements have more than one tunnel connecting them. What is the smallest value of N that guarantees that, no matter how the tunnels are drawn, it will be possible to travel between any two settlements?

21 *BAMO 2005*. There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these dirt roads. Prove that the government of Euleria may pave some of these dirt roads so that every city will have an odd number of paved roads leading out of it.

22 Show that a simple graph satisfying $e > v^2/4$ contains a triangle.

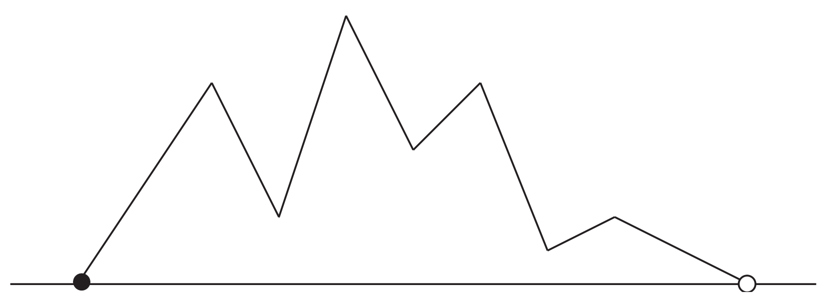
23 Show that if a graph is simple and satisfies

$$\sum \binom{d(v)}{2} > (m-1)\binom{v}{2},$$

then the graph contains $K_{2,m}$, where $m \geq 2$, $d(v)$ denotes the degree of vertex v, and the sum above is taken over all vertices v in the graph.

24 Show that, given a set of n points in the plane, the number of pairs of points at distance exactly 1 is at most $n^{\frac{3}{2}}\sqrt{2}+\dfrac{n}{4}$.

25 *The Two Men of Tibet*. Two men are located at opposite ends of a mountain range, at the same elevation. If the mountain range never drops below this starting elevation, is it possible for the two men to walk along the mountain range and reach each other's starting place, while always staying at the same elevation? Here is an example of a "mountain range." Without loss of generality, it is "piecewise linear," i.e., composed of straight line pieces. The starting positions of the two men is indicated by two dots.



26 Given a set of n points in the plane, the maximum possible number of pairs of points at distance greater than $1/\sqrt{2}$ is $\lfloor n^2/3 \rfloor$. Show that this maximum can be achieved.

27 A rectangle is tiled with smaller rectangles, each of which has at least one side of integral length. Prove that the tiled rectangle also must have at least one side of integral length.

Definition of Ramsey Numbers

We develop the Ramsey numbers in several steps, in increasing order of generality.

- Let K_n denote the *complete graph* on n vertices; i.e., the graph with n vertices with each pair joined by an edge.
- *The basic, or "edge" Ramsey numbers*. Define $R(m,n)$ to be the *minimum* integer N such that, if $N \geq R(m,n)$ and each edge of K_N is colored red or blue, then there must exist either a red K_m subgraph or a blue K_n subgraph. We can also call this number $R_2(m,n)$; the "2" refers to the fact that we are coloring edges. However, if there is no subscript, we assume that it is 2. For example, $R(3,3) = 6$.
- Next, we call a subset a u-subset if it contains u elements. Now we define $R_u(m,n)$ to be the minimum integer N such that, if $N \geq R_u(m,n)$ and each u-subset of a set with N elements is "colored" red or blue, then there must exist either an m-subset, all of whose u-subsets are red, or an n-subset, all of whose u-subsets are blue.
- Finally, we let the number of *colors* be arbitrary, and define $R_u(n_1, n_2, \ldots, n_c)$ as above, except that now we are using c colors to "color" the u-subsets. For example, what is $R_1(2,2,2,\ldots,2)$, where the parentheses contain t 2's? The answer is $t+1$. This is just the pigeonhole principle!

The theorems

Ramsey's Theorem (1930) The Ramsey numbers R as defined above exist (and are finite!)

Van der Waerden's Theorem (1927) If the integers $\mathbf{Z}$ are partitioned into finitely many sets, then one of these sets contains finite arithmetic progressions of arbitrary length.

Erdős-Szekeres Theorem (1935) For each n there exists $E(n)$ such that any set of $E(n)$ or more points in the plane (no three collinear) must contain a subset with n points that are the vertices of a *convex* n-gon.

Ramsey Theory Questions

1 Given six people, show that either three are mutual friends, or three are complete strangers to one another. (Assume that "friendship" is mutual; i.e., if you are my friend then I must be your friend.)

2 Seventeen people are at a party. It turns out that for each pair of people present, exactly one of the following statements is always true: "They haven't met," "They are good friends," or "They hate each other." Prove that there must be a trio (3) of people, all of whom are either mutual strangers, mutual good friends, or mutual enemies.

3 Ten people are in an elevator. Prove that either three know each other or four people are mutual strangers. Show that this is not necessarily true if we only have eight people.

4 Color the lattice points of the plane in two colors. Prove that there must be a rectangle (with sides parallel to the axes) each of whose vertices are the same color.

5 Prove that (with appropriate conditions on m, n, r)

(a) $R(m,n) \leq R(m,n-1) + R(m-1,n)$. Hint: induct on $m+n$.

(b) $R_u(m,n) \leq R_{u-1}\left(R_u(m,n-1), R_u(m-1,n)\right) + 1$.

(c) $R_u(n_1, n_2, \ldots, n_c) \leq R_u\left(R_u(n_1, \ldots, n_{c-1}), n_c\right)$.

This will establish Ramsey's Theorem!

6 Using the recurrence relations and constructions, show that

(a) $9 \leq R(3,4) \leq 10$.

(b) $14 \leq R(3,5) \leq 15$.

7 About 20 years ago, Erdős-Szekeres convex n-gon theorem was proven in a new way by an Israeli math student who was given the problem on an exam, but had missed the classes covering the material. The new method also uses Ramsey numbers, but instead of looking at quadruples, considers triples. Can you discover this method?