Problems about Squares

BMC, Jan. 5 2009

The following three problems are quite different, except that they all deal with squares in some way. But beyond that, working them out in full requires proving things about geometrical structures – proofs that are quite different from the standard "angle chasing" involved in a typical plane geometry problem.

See if you can work out the solutions and, if possible, justify their completeness with a proof.

1. It's easy to tile a square with 4 squares ("tile" means that we partition the large square into 4 smaller squares, no overlaps allowed).
    - Can you tile a square with 7 squares? They don't all have to have the same size.
    - How about 10? 6?
    - Can you determine precisely the set of numbers for which this is possible?
    - Can you prove that the impossible partition numbers really are impossible?
2. Consider four points located at the corners of the square. Look at all pairs of these points, and the distances between the two points in such a pair. You'll notice that there are six pairs, so six distances, but there are only two unique distances among those six: the length of the side of the square, and the length of its diagonal.
    - Can you find other configurations of four points in the plane with the property that there are just 2 distinct distances between pairs of these points?
    - Can you make a list you believe is exhaustive – covers every possible such configuration?
    - Can you *prove* that your list is exhaustive, meaning can you guarantee there aren't any other configurations you may have missed?
3. Once again consider the four corners of the squares, and imagine those are the locations of cities in the dessert. We'd like to build roads that connect those cities, meaning that using these roads we can get from any city to any other city among these four. However, roads are extremely expensive to build in this climate, so we'd like to minimize the total length of the roads.
    - Can you achieve this with roads of total length 3?
    - Can you achieve this with roads of total length *less* than 3?
    - How short can you make the roads?
    - Can you *prove* that no shorter road system is possible?