WHAT IS A TROPICAL LINE?

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1. Introduction

Tropical geometry is a fairly new invention and is a subject of much current research. There are a few good introductions, and the material here is a compilation from [K], [RGST], [MS], and [SS]. There is considerably more material here than was presented in the lecture, but students are encouraged to read and work through the later sections as well as the material actually covered.

2. Tropical arythmetic and algebra.

Definition 1. *Tropical semiring* is the real numbers $\mathbb{R}$ with operations of *tropical sum* $x \oplus y \ := \ \min(x,y)$ and *tropical product* $x \odot y \ := \ x+y$.

While $\mathbb{R}$ with usual operations is a ring, the tropical version is only a semi-ring, because there is no tropical subtraction (why?).

Exercise 2. Show that the *distributive property* holds, that is $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

Exercise 3. What plays a role of 0 in tropical arithmetic? What about the role of 1?

Example 4.

$$\begin{aligned} (x \oplus y)^3 &= (x \oplus y) \odot (x \oplus y) \odot (x \oplus y) \\ &= 0 \odot x^3 \ \oplus \ 0 \odot x^2 y \ \oplus \ 0 \odot x y^2 \ \oplus \ 0 \odot y^3. \end{aligned}$$

Exercise 5. Prove that *Freshman's Dream* holds for all powers in tropical arithmetic: $(x \oplus y)^n = x^n \oplus y^n$.

Exercise 6. Define *arctic semiring* operations as $x \boxplus y := \max(x,y)$ and $x \boxdot y := x+y$. as before (warning: this is not standard notation). Show that the resulting structure is isomorphic to the tropical semiring defined above.

Thus there are two possible equivalent definitions of tropical operations (one with max and one with min). At the moment there is no universal agreement on which one to use. We use the min-plus convention.

2.1. **Polynomials.** We let $x_1, x_2, \ldots, x_n$ be variables representing elements of the the tropical semiring. As usual a monomial is a product of finite number of these, which we can write in the usual manner:

Example 7. $x_2 \odot x_1 \odot x_3 \odot x_1 \odot x_4 \odot x_2 \odot x_3 \odot x_2 \quad = \quad x_1^2 x_2^3 x_3^2 x_4$.

Note that this uses the commutativity of multiplication (why?).

Exercise 8. Show that monomials represent linear functions $\mathbb{R}^n$ to $\mathbb{R}$ with nonnegative integer coefficients and all such functions arise from monomials.

Definition 9. A *tropical polynomial* is a function $\mathbb{R}^n$ to $\mathbb{R}$ which is given as a finite linear combination (that is a sum with coefficients) of (tropical) monomials.

Note that a polynomial is a *function* - it is a minimum of a set of linear functions. It can happen that for some monomials the value of this minimum never coincides with the value of that monomial. Then small change in the coefficient of that monomial does not change the function.

Example 10. We have $x^2 \oplus 1 \otimes x \oplus 2 = x^2 \oplus 10 \otimes x \oplus 2$.

Exercise 11. Why does this not happen in "classical" arithmetic? That is, why do different polynomials define different functions?

Exercise 12. Show that any function f from $\mathbb{R}^n$ to $\mathbb{R}$ represented by tropical monomial is piecewise linear with finitely many linear pieces with nonnegative integer coefficients, continuous and concave.

Exercise 13. Formulate and prove the converse to the previous exercise.

Exercise 14. Prove the *Tropical Fundamental Theorem of Algebra*: Every polynomial in one variable factors into a product of linear polynomials in a unique way (up to a permutation of factors).

The factorization of multivariate tropical polynomials into irreducible tropical polynomials is not unique.

Example 15.

$$\begin{aligned} & (0 \odot x \oplus 0) \odot (0 \odot y \oplus 0) \odot (0 \odot x \odot y \oplus 0) \\ = \; & (0 \odot x \odot y \oplus 0 \odot x \oplus 0) \odot (0 \odot x \odot y \oplus 0 \odot y \oplus 0). \end{aligned}$$

Developing an algorithm to find all irreducible factorizations of of a given tropical polynomial is an open problem.

3. A line in the tropical plane

We want to define a tropical line. First we try the usual definitions of the line.

Proposal 16. A line should be the solution set of one linear equation

$$a \odot x \oplus b \odot y \oplus c \quad = \quad a' \odot x \oplus b' \odot y \oplus c'.$$

This leads to lines that are one-dimensional in most cases, but can be two-dimensional. An alternative definition:

Proposal 17. A line is the span of two points a and b.

This is the set of the following points in $\mathbb{R}^2$ as the scalar λ ranges over $\mathbb{R}$: $\lambda a + (1-\lambda)b$.

This runs into an immediate problem: we have no subtraction in the tropical ring. There is a way around it: work in projective space, where all coordinates are defined up to an additive constant. Then the coordinates are (x, y, z) and a line is a span $\lambda \odot a \oplus \mu \odot b$ where λ and μ are any numbers in $\mathbb{R}$.

Exercise 18. (For those familiar with the projective plane.) Investigate what "lines" this definition produces. If you get stuck, see [RGST].

Proposal 19. A line is (usually) a graph of a linear function.

This has a nice feature of producing one-dimensional lines, but has some strange features: a graph from x to y is never a graph from y to x, and through any two points there is generally either one or the other type of line. In addition, two lines may fail to meet at a single point.

Additional requirements: Lines are one dimensional objects. Two general lines in the plane should meet at one point. Two general points should lie on one line.

How can we get tropical lines that satisfy this in a natural way?

3.1. **Tropical world as a degeneration of the classical world.** Consider the the plane of complex numbers $\mathbb{C}^2$, that is the set of all *pairs* of complex numbers. If you are not comfortable with complex numbers, you can think about positive real numbers everywhere. We will write complex coordinates as capital letters X, Y. Let's consider a pair of points P_1 and P_2 with coordinates X_1, Y_1 and X_2, Y_2 and a (complex) line L passing through these points. This line is given by the equation

$$\frac{X - X_1}{X_2 - X_1} = \frac{Y - Y_1}{Y_2 - Y_1}$$

.

We are interested in complex solutions of this equation. Each complex number is representable as $X = \varphi r$ where φ is a unimodular complex number, aka *phase*, and $r \geq$ is the *amplitude*. Supposing $r > 0$ we can write $r = t^x$ for a fixed real $t > 1$.

We can therefore set $P_i = (\varphi_i t^{x_i}, \psi_i t^{y_i})$ and the equation of L takes the form (as usual)

$$AX + BY + C = 0$$

with $A = \frac{1}{(X_2 - X_1)} = \frac{1}{(\varphi_2 t^{x^2} - \varphi_1 t^{x_1})}$, and similarly for coefficients B and C.

Now suppose that the phases φ_i, ψ_i and the amplitude exponents x_i y_i for the points P_i are fixed, while t changes. We get a family of lines $L(t)$ depending on a real parameter t.

We are interested in the behavior of this family as $t \mapsto \infty$. Note that for large t the coefficient $A(t)$ behaves like a function constt^α. Indeed, if $x_2 > x_1$ we get

$$A(t) = \frac{1}{(\varphi_2 t^{x^2} - \varphi_1 t^{x_1})} = \frac{t^{-x_2}}{\varphi_2 - \varphi_1 t^{x_1 - x_2}} \sim \frac{1}{\varphi_2} t^{-x_2}$$

So in this case $\alpha = -x_2$. When $x_1 < x_2$ we have a similar representation with $\alpha = -x_1$. Similar computations lead to $B \sim \text{const} t^{\beta}$, $C \sim \text{const} t^{\gamma}$. Setting in addition $X = \varphi t^x$ and $Y = \psi t^y$ we can rewrite the equation for L as

$$a t^{x+\alpha} + b t^{y+\beta} + c t^{\gamma},$$

where a, b and c are non-zero constants independent of t.

If all the exponents are different, one of the monomials will dominate the others, and the whole thing can not become zero. So for $(\varphi t^x, \psi t^y)$ to be a solution we need to have $max(x + \alpha, y + \beta, \gamma)$ achieved at least twice.

In the notation of the previous section, this is saying that the limit as t goes to infinity of the line $L(t)$ is the set where the function $f(x, y) = \alpha \odot x \oplus \beta \odot y \oplus \gamma$ switches between its domains of linearity.

Definition 20. A *line in the tropical plane* is a set of all (x, y) where the maximum of $x + \alpha, y + \beta, \gamma$ is achieved at least twice.

Then what we did in this section can be reinterpreted as follows:

Putting $\mathbb{C}^* = \mathbb{C} \setminus 0$ consider the map

$$\text{Log}_t : (\mathbb{C}^*)^2 \mapsto \mathbb{R}^2$$

given by $(X, Y) \mapsto (x, y), x = \log_t |X|, y = \log_t |Y|$

Then the image of $L(t)$ under $(Log)_t$ is contained for large t in some ϵ neighborhood of a tropical curve, and the size ϵ of this neighborhood can be made arbitrarily small when t goes to infinity.

The image of a complex line under Log_t is called an *amoeba*. Interior points of the amoeba have two preimages, while the boundary ones have one. The complex line can be imagines as a result of glueing two copies of the amoeba along their common boundary. The resulting "pair of pants" is topologically equivalent to a sphere with three points removed. These points correspond to the intersection points of $L(t)$ with the coordinate axes and to the "point at infinity" of L.

We can now see that tropical lines defined in this manner satisfy our requirements.

Exercise 21. Check this.

Moreover, the lines do not really have to be generic to have a unique intersection! Indeed, consider some degenerate configuration of lines. Perturb first one of the lines slightly. Now it has a unique intersection with the second one. Note that there is a unique limit for this intersection when the perturbation is small (find it!). So any two lines have a unique preferred intersection point.

Exercise 22. Show that any two points define a unique preferred line.

3.2. **Degeneration in arythmetic.** The "look at log_t as t goes to infinity" approach to tropical mathematics reproduces the tropical semiring as weell.

Indeed, consider $\mathbb{R}$ with the operations

$$x \times_t y = \log_t(t^x t^y) = x + y$$

$$x +_t y = \log_t(t^x + t^y),$$

where $t > 1$ is a parameter.

When we identify $\mathbb{R}$ with $\mathbb{R}_{>0}$ via $x \mapsto t^x$ the operations above coincide with usual addition and multiplication on $\mathbb{R}_{>0}$. However, in the limit as t goes to infinity, these operations become the tropical operations!

Exercise 23. Check this.

Exercise 24. Deduce commutativity, associativity and distributivity of tropical operations from this definition.

4. Curves in a tropical plane.

This definition of a tropical line readily generalizes to a definition of a tropical curve. Consider a tropical polynomial, and the corresponding set of points where it switches between the domains of linearity. Pick one of the segments of this "domain of nonlinearity". Suppose that for that segment one the one side of it the minimal monomial is $a_ij \odot x^i \odot y^j = ix + jk + a_ij$ and on the other side the monomial $a_{\hat{i}\hat{j}} \odot x^{\hat{i}} \odot y^{\hat{j}} = \hat{i}x + \hat{j}y + a_{\hat{i}\hat{j}}$.

Then the line containing this segment is given by equation $(i - \hat{i})x + (j - \hat{j})y + (a_ij - a_{\hat{i}\hat{j}}) = 0$

We define a *multiplicity* of such a segment is the GCD of $(i - \hat{i})$ and $(j - \hat{j})$. This depends on the polynomial, and is not possible to read off from the segment itself.

Definition 25. A tropical curve associated to a given tropical polynomial is a set of all points of nonlinearity of the polynomial together with all the weights of the edges.

This set is a graph embedded in the plane.

Lemma 26. *Tropical curves associated to a polynomial of degree d has the following properties:*

1) Every edge has a rational slope

2) At each vertex the following balancing condition holds*: Let v_i be the vector along an edge emanating from a given vertex and equal to the shortest integer vector in that direction multiplied by the weight of the edge. Then $\Sigma v_i = 0$.*

3) There are $3d$ *infinite edges (with multiplicity), d directed up, d directed to the right and d directed down and left with slope 1.*

The first property is fairly obvious. Let's prove the second. Suppose near a given vertex the minimum is attained by functions $i_1x + j_1y + a_{i_1j_1}, \ldots i_rx + j_ry + a_{i_rj_r}$. Then we have the obvious equalities

$$(i_2 - i_1) + \ldots + (i_r - i_{r-1}) + (i_1 - i_r) = 0$$
$$(j_2 - j_1) + \ldots + (j_r - j_{r-1}) + (j_1 - j_r) = 0$$

but the vector v_s in the balancing condition is obtained from the vector $(i_{s+1} - i_s, j_{s+1} - j_s)$ by rotation by 90 degrees.

Exercise 27. Prove property 3.

Theorem 28. *Tropical polynomial can be reconstructed from its curve up to an additive (i.e. tropically multiplicative) constant. Moreover, any weighted graph in the plane satisfying conditions (1-3) of the above lemma is a tropical curve associated to some polynomial of degree d.*

Exercise 29. Prove this.

Note that for two polynomials of degree d_1 and d_2 such the corresponding curves intersect only in the infinite parts, the number of intersection points (with appropriate multiplicities) is $d_1 d_2$. This is an instance of a tropical Bezout's theorem, and is indeed true for any two tropical curves. See [RGST] for a fairly accessible proof.

5. Tropical mathematics in other areas of mathematics

5.1. **Assignment problem.** Suppose we are given n tasks to perform and n workers who can perform them. Each worker will charge a different amount for each task. Our goal, then, is to figure out the minimal total of the work.

Suppose that the cost for the i'th worker to perform j'th task is $a(ij)$.

Then what we want to compute is $\min(a(1, \sigma(1)) + a(2, \sigma(2)) + \ldots + a(n, \sigma(n)))$

over all the permutations σ of numbers $1, \ldots, n$. This is nothing else but the tropical determinant of the matrix of $a(i, j)$'s!

The assignment problem is one of the oldest problems in optimization and it is solved by a so-called Hungarian algorithm.

Let's see how it works. (This proof can be found on Wikipedia page for teh Hungarian Algorythm and is included to make these notes more self-contained). First of all we reformulate this problem as follows - we consider a complete bipartite graph with the set V of n worker vertrices and set T of n task vertices. Each vertex in V is connected to each vertex in T by an edge weighted by the cost for corresponding worker to perform the corresponding task. We want to find a perfect matching with smallest total cost.

Let us call a function $y : (V \cup T) \mapsto \mathbb{R}$ *potential* if

$$y(i) + y(j) \leq a(i, j)$$

for each $i \in V, j \in T$. The value of potential y is $\sum_{v \in V \cup T} y(v)$. Hence the cost of each perfect matching is at least the value of each potential. The Hungarian method finds a perfect matching and a potential with equal cost/value which proves the optimality of both. In fact it finds a perfect matching of *tight edges*: an edge ij is called tight for a potential y if $y(i) + y(j) = a(i, j)$. Let us denote the subgraph of tight edges G_y. The cost of a perfect matching in G_y (if there is one) equals the value of y.

During the algorithm we maintain a potential y and an orientation of G_y (denoted by $\overrightarrow{G_y}$) which has the property that the edges oriented from T to V form a matching M. Initially, y is 0 everywhere, and all edges are oriented from V to T (so M is empty). In each step, either we modify y' so that its value increases, or modify the orientation to obtain a matching with more edges. We maintain the invariant that all the edges of M are tight. We are done if M is a perfect matching.

In a general step, let $R_V \subseteq V$ and $R_T \subseteq T$ be the vertices not covered by M (so R_V consists of the vertices in V with no incoming edge and R_T consists of the vertices in T with no outgoing edge in $\overrightarrow{G_y}$). Let Z be the set of vertices reachable in $\overrightarrow{G_y}$ from R_V by a directed path in $\overrightarrow{G_y}$ (and hence only following edges that are tight).

If $R_T \cap Z$ is nonempty, then reverse the orientation of a directed path in $\overrightarrow{G_y}$ from R_V to R_T. Thus the size of the corresponding matching increases by 1.

Exercise 30. Check that the set of edges from T to V in the new $\overrightarrow{G_y}$ is still a matching.

If $R_T \cap Z$ is empty, then let $\Delta := \min\{c(i,j) - y(i) - y(j) : i \in Z \cap V, j \in T \setminus Z\}$. This is a measure of slack in our potential.

Exercise 31. Show that there are no tight edges between $Z \cap V$ and $T \setminus Z$

This means that Δ is positive.

Increase y by Δ on the vertices of $Z \cap V$ and decrease y by Δ on the vertices of $Z \cap T$.

Exercise 32. The resulting y is still a potential.

The graph G_y changes, but it still contains M (why?). We orient the new edges from V to T. By the definition of Δ the set Z of vertices reachable from R_V increases (note that the number of tight edges does not necessarily increase).

We repeat these steps until M is a perfect matching, in which case it gives a minimum cost assignment.

The running time of this version of the method is $O(n^4)$: M is augmented at most n times, and in a phase where M is unchanged, there are at most n potential changes (since Z increases every time). The time needed for a potential change is $O(n^2)$.

One can think of a slightly more general problem, where we are allowed to assign parts of each task to different workers, but each worker must complete a total of one task (and of course each task can only be split into parts adding to 1). The goal then, is to find a set of weights $w(i,j)$ for each edge, $w(i,j) \geq 0$ such that for each j we have $\Sigma_j w(i,j) = 1$ and for each i, similarly, $\Sigma_i w(i,j) = 1$ and so that the total cost $\Sigma_(i,j) w(i,j) a(i,j)$ is minimal. This is a program of *linear programming*: it can be written in a standard form as "minimize $C^T \dot{X}$ subject $X \geq 0$ and $AX = B$" where in this case C is a n^2 vector of costs, X is the n^2 long vector of weights, A is the n^2 rows by $2n$ columns incidence matrix of edges and vertices, and B is the $2n$ long vector of all 1's.

Then the *dual problem* is to maximize $B^T Y$ where Y is a function on the space of constraints (i.e. vertices), subject to $Y \geq 0$ and $A^T Y \geq C$, that is the sum of values of

Y for each edge is bigger than the cost. Thus we arrive exactly at the potential in the Hungarian algorithm.

It is a general fact from linear programming that any feasible value for the dual problem (that is, an assignment of Y variables) provides a bound on the original problem.

Exercise 33. Prove this.

Moreover, the assignment problem can be reformulated as a problem of finding a minimal cost flow in a directed graph, for which the integrality theorem (see for example [AOM]) states that any optimal solution is integral, meaning that the weights $w(i,j)$ must be integral. As the weights here are bounded between 0 and 1 they must *be* 0 or 1. So there is no difference between this problem and the one we started with originally.

Exercise 34. Formulate the assignment problem as a minimal cost graph flow problem.

Exercise 35. Prove the integrality for optimal solutions of the assignment problem directly.

5.2. **Phylogenetics.** An important problem in computational biology is to construct a *phylogenetic tree* from distance data involving n taxa. These taxa might be organisms or genes, each represented by a DNA sequence.

From sequence data, computational biologists infer the distance between any two taxa. There are various algorithms for carrying out this inference. They are based on statistical models of evolution. The problem of phylogenetics is to construct a tree with edge lengths which represent this distance matrix, provided such a tree exists.

In general, considering n taxa, the *distance* between taxon i and taxon j is a positive real number d_{ij} which has been determined by some bio-statistical method. So, what we are given is a real symmetric $n \times n$-matrix

$$D \quad = \quad \begin{pmatrix} 0 & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{12} & 0 & d_{23} & \cdots & d_{2n} \\ d_{13} & d_{23} & 0 & \cdots & d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1n} & d_{2n} & d_{3n} & \cdots & 0 \end{pmatrix}.$$

We may assume that D is a *metric*, i.e., the triangle inequalities $d_{ik} \leq d_{ij} + d_{jk}$ hold for all i, j, k. This can be expressed by tropical matrix multiplication:

Exercise 36. The matrix D represents a metric if and only if $D \odot D = D$.

We say that D is a *tree metric* if there exists a tree T with n leaves, labeled $1, 2, \ldots, n$, and a positive length for each edge of T, such that the distance from leaf i to leaf j equals d_{ij} for all i, j. Tree metrics occur naturally in biology because they model an evolutionary process that led to the n taxa.

Most metrics D are not tree metrics. We want to see which ones are, and it turns out we can state a simple criterion for a metric to be a tree metric in tropical terms.

Let $X = (X_{ij})$ be a symmetric matrix with zeros on the diagonal whose $\binom{n}{2}$ distinct off-diagonal entries are unknowns. For each quadruple $\{i, j, k, l\} \subset \{1, 2, \ldots, n\}$ we consider the following tropical polynomial of degree two:

$$p_{ijkl} \quad = \quad X_{ij} \odot X_{kl} \;\oplus\; X_{ik} \odot X_{jl} \;\oplus\; X_{il} \odot X_{jk}. \tag{1}$$

This polynomial is the *tropical Grassmann-Plücker relation.* It defines a hypersurface $\mathcal{H}(p_{ijkl})$ in the space $\mathbb{R}^{\binom{n}{2}}$. The *tropical Grassmannian* is the intersection of these $\binom{n}{4}$ hypersurfaces.

Fact 37. *A metric D on $\{1, 2, \ldots, n\}$ is a tree metric if and only if its negative $X = -D$ is a point in the tropical Grassmannian $Gr_{2,n}$.*

The statement is a reformulation of the *Four Point Condition* in phylogenetics, which states that D is a tree metric if and only if, for all $1 \leq i < j < k < l \leq n$, the *maximum* of the three numbers $D_{ij} + D_{kl}$, $D_{ik} + D_{jl}$ and $D_{il} + D_{jk}$ is attained at least twice.

Exercise 38. Prove that the four point condition is necessary and sufficient for D to be a tree metric.

For $X = -D$, this means that the *minimum* of the three numbers $X_{ij} + X_{kl}$, $X_{ik} + X_{jl}$ and $X_{il} + X_{jk}$ is attained at least twice, or, equivalently, $X \in \mathcal{H}(p_{ijkl})$.

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