Berkeley Math Circle
BAMO Preparation – Ivan Matić

1. If n is an integer, prove that the number
$$1+n+n^2+n^3+n^4$$
is not divisible by 4.

2. Do there exist positive integers x, y, and z such that $x^{2009}+y^{2009}=z^{2010}$? Explain your answer.

3. A square grid of 16 dots contains the corners of nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of 14 squares whose sides are parallel to the sides of the grid. What is the smallest possible number of dots you can remove so that, after removing those dots, each of the 14 squares is missing at least one corner?

   Justify your answer by showing both that the number of dots you claim is sufficient and by explaining why no smaller number of dots will work.

4. A magical tree contains 2005 green and 2006 red apples. Every time a child climbs the tree, he (she) eats 2 apples. After that a miracle happens: when the child takes 2 apples of the same color, one red apple grows on the tree; when the child takes 2 apples of different colors, one green apple grows on the tree.

   What will be the color of the last apple? Why?

5. A frog is jumping on a 8×8 chessboard. At each step the frog jumps from one unit square to one of the squares that is adjacent to the previous position of the frog (squares are *adjacent* if they share an edge). Is it possible for frog to start from the lower-left corner of the chessboard, visit each unit square exactly once, and finish its trip at the upper-right corner of the chessboard? Justify your answer!

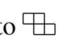
6. Given three squares of dimensions 2×2, 3×3, and 6×6, choose two of them and cut each into 2 figures, such that it is possible to make another square from the obtained 5 figures.

7. The *Fibonacci sequence* is the list of numbers that begins 1, 2, 3, 5, 8, 13 and continues with each subsequent number being the sum of the previous two. Prove that when the first n elements of the Fibonacci sequence are alternately added and subtracted, the result is an element of the sequence or the negative of an element of the sequence. For example,
$$1-2+3-5=-3,$$
and 3 is an element of the Fibonacci sequence.

8. Call a year ultra-even if all of its digits are even. Thus 2000, 2002, 2004, 2006, and 2008 are all ultra-even years. They are all 2 years apart, which is the shortest possible gap. 2009 is not an ultra-even year because of the 9, and 2010 is not an ultra-even year because of the 1.

   (a) In the years between the years 1 and 10000, what is the longest possible gap between two ultra-even years? Give an example of two ultra-even years that far apart with no ultra-even years between them. Justify your answer.

   (b) What is the second-shortest possible gap (that is, the shortest gap longer than 2 years) between two ultra-even years? Again, give an example, and justify your answer.

9. Determine the greates number of figures congruent to  that can be placed in a grid 2005×2005 (without overlaping) such that each figure covers exactly 4 unit squares?

10. Initially, there is one pile with 100 coins in it. A player plays the following game: In each step she chooses one of the piles that have more than one coins, splits the pile into two, and writes on the blackboard the product of the number of coins in the smaller piles. The game ends when all the remaining piles have one coin each. In the end of the game, the sum of all the numbers written on the blackboard is calculated. What is the maximal sum the player can obtain?