Berkeley Math Circle
AIME Preparation – Ivan Matić

1. Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is m/n, where m and n are relatively prime positive integers. Find $m+n$.

2. Let $A_1, \ldots, A_{12}$ be the vertices of a regular dodecagon. How many distinct squares in the plane of the dodecagon have at least two vertices in the set $\{A_1, \ldots, A_{12}\}$?

3. Let S be the set of points whose coordinates x, y, and z are integers satisfying $0 \leq x \leq 2$, $0 \leq y \leq 3$, $0 \leq z \leq 4$. Two distinct points are randomly chosen from S. The probability that the segment they determine also belongs to S is m/n where m and n are relatively prime. Find $m+n$.

4. Let S be the set $\{1, 2, 3, \ldots, 10\}$. Let n be the number of sets of two non-empty disjoint subsets of S. Find the remainder obtained when n is divided by 1000.

5. A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

6. A basketball player has a constant probability of 0.4 of making any given shot, independently of previous shots. Let a_n be the ratio of shots made to shots attempted after n shots. The probability that $a_{10} = 0.4$ and $a_n \leq 0.4$ for all n such that $1 \leq n \leq 9$ is given to be $p^a q^b r / s^c$ where p, q, r, and s are prime numbers and a, b, and c are positive integers. Find $(p+q+r+s)(a+b+c)$.

7. The numbers 1, 2, 3, 4, 5, 6, 7, and 8 are randomly written on the faces of a regular octahedron so that each face contains a different number. The probability that no two consecutive numbers, where 8 and 1 are considered to be consecutive, are written on faces that share an edge is m/n where m and n are relatively prime positive integers. Find $m+n$.

8. On a given 2008×2008 chessboard, each unit square is colored in a different color. Every unit square is filled with one of the letters A, I, M, E. The resulting board is called *harmonic* if every 2×2 subsquare contains all four different letters. How many harmonic boards are there?

9. Forty teams play a tournament in which every team plays every other team exactly once. No ties occur, and each team has a 50% chance of winning any game it plays. The probability that no two teams win the same number of games is $m/2^n$ where m is odd. Find n

10. Define a domino to be an ordered pair of distinct positive integers. A proper sequence of dominos is a list of distinct dominos in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not both appear for any i and j. Let D_{40} be the set of all dominos whose coordinates are no larger than 40. Find the length of the longest proper sequence of dominos that can be formed using the dominos of D_{40}.

11. Determine the number of bijections (i.e. permutations) $f : \{1, 2, \ldots, 2n\} \to \{1, 2, \ldots, 2n\}$ for which there exists an element $a \in \{1, 2, \ldots, 2n\}$ such that the set $\{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$ has exactly n elements.

12. n lines are given in a plane such that each two of them intersect but no three intersect at the same point. For which values of n is it possible to label the intersection points of these lines with numbers $1, 2, \dots, n-1$ such that on each of the lines each of the numbers appears exactly once?

13. 2009 different points are chosen in the plane and each of them is painted in either red or green. If every unit circle with green center has exactly two red points, what is the biggest possible number of green points?

14. In each cell of $n \times n$ ($n \geq 2$) there is one integer. All rows of this table are different. Prove that there exists a column after whose removal the remaining table satisfies that all of its rows are different.

15. Let $x_1, x_2, \dots, x_n$ be integers each of which is either 1 or -1. If $x_1x_2 + x_2x_3 + \cdots + x_nx_1 = 0$ prove that n is divisible by 4.

16. 6 different numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are chosen at random (all outcomes have the same probabilities). These numbers are painted in green and arranged in the increasing order $g_1 < g_2 < g_3 < g_4 < g_5 < g_6$. The remaining 6 numbers are painted in red and arranged in the decreasing order $r_1 > r_2 > r_3 > r_4 > r_5 > r_6$. Let X be a random variable defined as $X = \sum_{i=1}^{6} |r_i - g_i|$. Calculate $E(X)$.

17. A box contains one red and m blue balls (m is a positive integer). Bart and Lisa alternatively take the balls from the box without looking. Bart starts the game and in his first move he takes one ball at random from the box. If he picks the red one – he is the winner. Otherwise, Lisa takes a ball at random from the box (notice that now the box contains one red and $m-1$ blue balls). If she picks the red ball – she is the winner. If she takes the blue one, then it is Bart's turn again, and the game continues until someone takes the red ball. For each value of m determine who has a bigger chance for winning: Bart or Lisa.

18. 100 different numbers from the set $\{1, 2, \dots, 199\}$ are painted in green (each number has equal chance to be painted). What is the probability that the sum of no two green numbers is equal to 199 or 200? (i.e. what is the probability that none of the equations $x + y = 199$ and $x + y = 200$ have completely green solutions?)

19. Define a domino to be an ordered pair of distinct positive integers. A proper sequence of dominos is a list of distinct dominos in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not both appear for any i and j. Let D_{40} be the set of all dominos whose coordinates are no larger than 40. Find the length of the longest proper sequence of dominos that can be formed using the dominos of D_{40}.