Again and Again: Iteration

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Warmup
The seven-day-week and the names of the days became established in the first century of the common era. The names have an astrological significance. Using the "Chaldean" ordering, which was also used by the second century astronomer Cladius Ptolemy, the order of the heavenly bodies from the earth were:

Moon, Mercury, Venus, Sun, Mars, Jupiter, and Saturn.

If we assign the first hour of the first day to the Sun, the second hour to Venus, the third to Mercury, and so on, skipping from the Moon back out to Saturn, since there are 24 hours in a day, we find the first hour of the second day belongs to the Moon. Continuing, we find the first hour of the third day belongs to Mars.

1. Using the Germanic identifications:

Woden/Odin=Mercury, Frigg=Venus, Tiw/Tyr=Mars, and Thor=Jupiter,

show that this system does indeed give the standard days of the week if the day is named after the assignment of its first hour.

2. What would the days of the week be using the following "Egyptian" ordering used by Plato and Aristotle?

Moon, Sun, Mercury, Venus, Mars, Jupiter, and Saturn

3. What would the days of the week be in the Chaldean ordering if there were 20 hours in a day?

4. The true ordering of the solar system was introduced by the cleric Nicolaus Copernicus. For a 24 hour day, what should the days of the week be called using the modern ordering?

Sun, Mercury,Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune

Example of Iteration

In Proposition 2, Book 7 of the *Elements*, Euclid gives the following method for finding the greatest common divisor of two given whole numbers:

i) Divide the larger by the smaller to get the remainder.
ii) If the remainder is not 0, repeat step **(i)** with the smaller number and the remainder as the two numbers.
iii) If the remainder is 0, the smaller number is the GCD.

5. Use Euclid's algorithm to find the GCD of the following pairs of numbers:

a)12, 3 **b)**24, 9 **c)** 165, 462 **d)**3198, 3003

6. In your own words, why does this method work?

Two More Examples of Iteration

Cuneiform tablets show the Babylonians had the following algorithm for calculating square roots before 2000BCE:

i) Guess an approximation for the square root of the number.
ii) Get a better approximation by averaging together the approximation and the quotient of the number by the approximation.
iii) Repeat step **(ii)** with the better approximation.

7. Why does this method work if your first guess is already correct?

The Pythagoreans had the following algorithm for calculating square roots, which Plato makes reference to:

i) Guess a fraction for an approximation for the square root of the number.
ii) Get a better approximation by adding the numerator and denominator to get the new denominator and the number times the denominator plus the numerator to get the new numerator.
iii) Repeat step **(ii)** with the better approximation.

8. Why does this method work if your first guess is already correct?

9. Use both methods to calculate the square roots of the following numbers accurate to four decimal places starting with a guess of 1:

a)2 **b)**$\frac{1}{3}$ **c)** 10

10. Which method is better? Why?

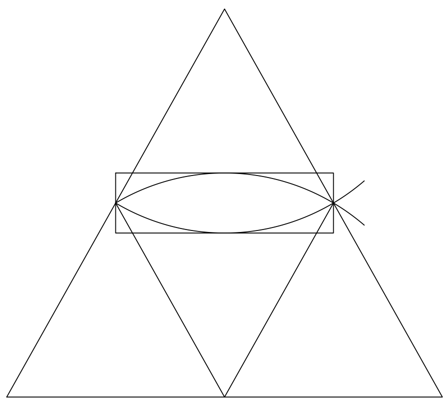

Figure 1: *Vesica Piscis*

An Ancient Puzzle

The book traditionally called the *Gospel of Yohanan* was written about 100CE. Here is an excerpt from the epilogue:

Shimon Kefa, Toma, Natanel, and the sons of Zavdai went out fishing and all that night caught nothing. At daybreak, Yeshua was standing on the beach. He said to them, "Children, have you any fish?"

"No," they answered him.

And he said to them, "Cast the net in the waters on the right side of the ship and you will find some." So they cast, and they were not strong enough to haul it back in because of the swarm of fish. They came in–they were not far from land, about a hundred yards away–dragging the net full of fish. When they came out on the shore, they saw a charcoal fire and bread.

Yeshua said to them, "Now bring some of the fish you caught." So Shimon Kefa went on board and dragged the net onto the land, filled with fish, a hundred fifty-three of them, yet with so many the net did not break.

The puzzle is the significance of the number 153. The 4th century translator Sophronius Eusebius Hieronymus thought this represented the number of species of fish, but there is no corroboration for this view among ancient authors. To solve this problem, look at the stylized fish above known as the *vesica piscis*. Using the Pythagorean theorem, the ratio of the length to the height of the body of the fish is $2+\sqrt{3}$. Our problem is then to find a connection between this irrational number and 153.

Another Example of Iteration
A continued-fraction is an expression where the denominators have fractions. For example, consider the sequence of continued fractions:

$$1+\frac{1}{1}$$

$$1+\cfrac{1}{1+\cfrac{1}{1}}$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}$$

11. Simplify the preceding expressions. Do you recognize a pattern?

Truncated continued fractions give excellent rational approximations to irrationals and to rationals with larger denominators. Here is an algorithm for calculating a continued fraction representation of a number:

i) Take the integer part of the number.
ii) Take the reciprocal of the decimal part and use that as the new number in step **(i)**.

12. Use the preceding algorithm to find continued fraction forms for the following:

a) $\frac{23}{7}$ **b)** $\frac{3157}{221}$ **c)** $\sqrt{2}$ **d)** $2+\sqrt{3}$

Note the preceding algorithm quickly breaks down on a calculator since we are only starting with eight digits of accuracy. There is a way to get the continued fraction exactly for an expression of the form of a number plus or minus a square root.

Example
To find a continued fraction for $1+\sqrt{2}$, notice that

$$\left(\left(1+\sqrt{2}\right)-1\right)^2=2$$

so

$$\left(1+\sqrt{2}\right)^2-2\left(1+\sqrt{2}\right)+1=2$$

and

$$\left(1+\sqrt{2}\right)^2 = 2\left(1+\sqrt{2}\right)+1$$

Dividing through by $1+\sqrt{2}$, we have

$$1+\sqrt{2} = 2 + \frac{1}{1+\sqrt{2}}$$

so

$$1+\sqrt{2} = 2 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}$$

and, by iteration,

$$1+\sqrt{2} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$$

13. Using the preceding procedure, find a continued fraction for the golden ratio $\tau = \dfrac{1+\sqrt{5}}{2}$.

14. To solve the puzzle, find a continued fraction for $1+\sqrt{3}$ using the preceding procedure. Does 153 appear if one instead uses the procedure on $2+\sqrt{3}$?

Dynamical Systems

Consider a sealed biosphere with algae. There is a limited amout of nutrition in the sphere, which can either be in solution or in the algae. Suppose the algae has a daily reproductive cycle and consider the following model:

i) Each unit of algae that matches a unit of solution doubles the next day.
ii) Each unit of algae that does not match a unit of solution dies and becomes a unit of solution.

15. Follow the rules with the starting values

a) algae=1, solution=7 **b)** algae=6, solution=3 **c)** algae=1, solution=6

16. Find a formula for the amount of algae tomorrow given the amount today if the total nutrition is 10.

Dynamical systems can be studied using spider diagrams:

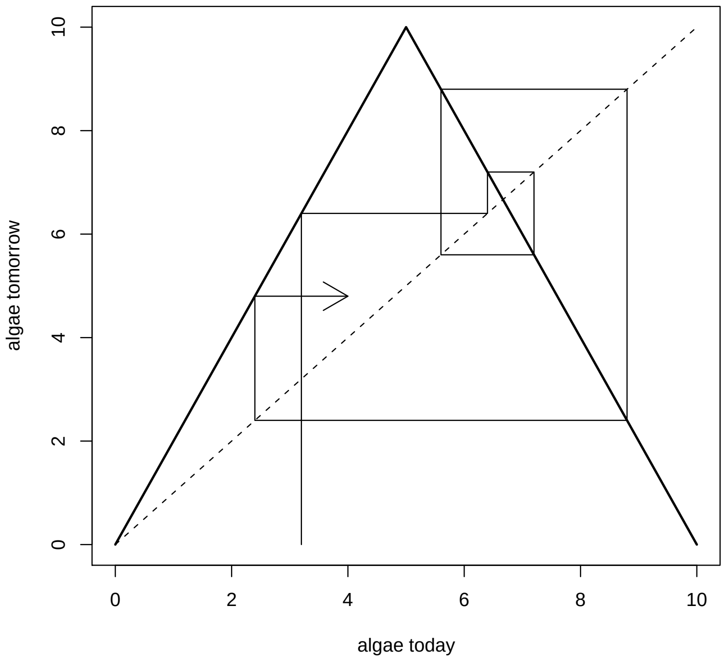

Figure 2: *Spider Diagram for Biosphere*

17. What are the fixed points for the biosphere model? Are they stable?

18. What happens if one starts with algae=3 if the total nutrition is 10? How can one make a spider diagram for cycles of length 2? of length 3?

19. Make spider diagrams for the Babylonian and Pythagorean methods of finding square roots.

20. What happens if one tries to find the continued fraction for $2 - \sqrt{3}$ using the procedure given? Make a spider diagram to help understand what is happening.