Winter Bulgarian Mathematical Competitions, 1998

Berkeley Math Circle

Advanced Session

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Problem 8.1. (a) If a, b, $c \in \mathbb{R}$ such that $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$, prove that $a = b = c$.

(b) Find the sum $x + y$ if $\frac{x}{3y} = \frac{y}{2x - 5y} = \frac{6x - 15y}{x}$ and $-4x^2 + 36y - 8$ is maximal.

Sava Grozdev

Problem 8.2. Let BE and CF be the altitudes of an acute $\triangle ABC$ with $\angle BAC = 45°$ and H, M, and K be its orthocenter and the midpoint of BC and AH, respectively.

(a) Prove that $MEKF$ is a square.
(b) Prove that the intersection point of the diagonals of $MEKF$ is the midpoint of OH, where O is the circumcenter of $\triangle ABC$.
(c) Find the length of EF if the circumradius of $\triangle ABC$ is 1.

Christo Lesov

Problem 8.3. Consider any 1998 points in the plane so that among any 17 of them there are 11 lying inside a circle of diameter 1. Find the least number of circles of diameter 2 sufficient to cover all 1998 points. (We say that a circle covers some points in the plane if all of them lie inside the circle or on its circumference.)

Sava Grozdev

Problem 9.1. Find all functions $f(x) = x^2 - ax + b$ with integer coefficients such that $|f(m)| = |f(n)| = |f(p)| = 7$ for some distinct integers $m, n, p \in [1, 9]$.

Lyubomir Davidov

Problem 9.2. Let A_1, B_1, and C_1 be the points on the sides BC, CA, and AB of $\triangle ABC$ such that $AB_1 = C_1B_1$ and $BA_1 = C_1A_1$. Denote by D the symmetric point of C_1 with respect to A_1B_1 ($D \neq C$). Prove that the line CD is perpendicular to the line passing through the circumcenters of $\triangle ABC$ and $\triangle A_1B_1C$.

Ljubomir Davidov, Peter Boyvalenkov

Problem 9.3. Each of the numbers 1,2,..., 1998 is taken 9 times and is written in a cell of a rectangular table with 9 rows and 1998 columns so that the difference of any two numbers lying in the same column is not greater than 3. Find the maximal possible value of the least sum amongst all 1998 sums formed by the numbers lying in the same column.

Sava Grozdev

Problem 10.1. Find all real numbers a for which the equation $x^3 - 3x^2 + (a^2 + 2)x - a^2 = 0$ has three distinct real roots x_1, x_2 and x_3 such that $\sin\left(\frac{2\pi}{3}x_1\right), \sin\left(\frac{2\pi}{3}x_2\right)$, and $\sin\left(\frac{2\pi}{3}x_3\right)$ form (in some order) an arithmetic progression.

Rumen Kozarev

Problem 10.2. For any point C on a circle, two points A and B are chosen articlockwise on it so that if $\angle CAB = \alpha$ and $\angle CBA = \beta$ then

$$2\cos\left(\frac{\alpha}{2} + \beta\right) = \sin\left(\frac{\alpha}{2} - \beta\right).$$

Prove that the angle bisectors of $\angle CAB$ pass through a fixed point.

Emil Kolev

Problem 10.3. Let n be a positive integer. Find the number of sequences $a_1, a_2, \ldots, a_{2n}$ such that $a_i = \pm 1$ and

$$\left|\sum_{i=2k-1}^{2} la_i\right| \leq 2 \text{ for all } 1 \leq k \leq l \leq n.$$

Ljubomir Davidov, Emil Kolev

Problem 11.1. Consider the function

$$f(x) = \sqrt{x} + \sqrt{x-4} - \sqrt{x-1} - \sqrt{x-3} \text{ for } x \geq 4.$$

(a) Find $\lim_{x\to\infty} f(x)$.
(b) Prove that $f(x)$ is an increasing function.
(c) Find the number of real roots of the equation $f(x) = a\sqrt{\frac{x-3}{x}}$ where a is a real number.

Oleg Mushkarov, Nikolai Nikolov

Problem 11.2. The convex quadrilateral $ABCD$ is inscribed in a circle with center O, and let $E = AC \cap BD$. Prove that if the midpoints of AD, BC, and OE are collinear then either $AB = CD$ or $\angle AEB = 90°$.

Nikolai Nikolov

Problem 11.3. Let $\{a_m\}_{m=1}^{\infty}$ be the sequence of positive integers having only even digits in their decimal representations: $a_1 = 2$, $a_2 = 4$, $a_3 = 6$, $a_4 = 8$, $a_5 = 20, \ldots$. Find all positive integers m such that $a_m = 12m$.

Oleg Mushkarov