Problem List: Euler Number
September 29 / September 30, 2009

Most of these problems are taken from the book Circle in a Box, by Sam Vandervelde.

1. In a certain small country, there are villages, expressways, and fields. Expressways only lead from one village to another and do not cross one another, and it is possible to travel from any village to any other village along the expressways. Each field is completely enclosed by expressways and villages. If there are 70 villages and 100 expressways, how many fields are there?
2. Find a way to position four points on a sheet of paper so that when every pair of points is joined by a straight line segment, none of the segments intersect. (There will be six segments.)
3. Plot five points on a sheet of paper and draw a straight line segment connecting every pair of points. How many edges are needed? Can you draw the lines so that none of them cross?
4. Prove that it is not possible to draw the lines in #3 so that none of them cross. (Hint: first show that $2E >= 3F$, thinking about the shapes of the faces.) What if the line segments are allowed to curve, but still can't cross?
5. Three houses need to be connected with lines to three utilities (gas, water, and electricity), in such a way that the lines can curve but can never cross. Can it be done or is it impossible? Prove it!
6. A certain polyhedron is built entirely from triangles, in such a way that 5 faces meet at each vertex. Prove that it has to have 20 faces. (Hint: first deduce that $3F = 2E$ and $3F = 5V$)
7. A certain polyhedron is made of squares, hexagons, and decagons (which are 10-sided polygons), in such a way that one square, one hexagon, and one decagon meet at each vertex. How many vertices does it have?
8. A soccer ball is made up of pentagons and hexagons in such a way that three polygons meet at each vertex. How many pentagons must there be?
9. Another polyhedron is built entirely from triangles, in such a way that 4 faces meet at each vertex. Prove that it has to have 8 faces. What about if 3 faces meet at each vertex? What happens if 6 faces meet at each vertex?
10. A polyhedron is called regular if all its faces are the same regular polygons (for example, all equilateral triangles or all squares) and if the same number of faces meet at each vertex (for example, 3 faces meet at each vertex). For example, the cube, the tetrahedron, the octahedron, the dodecahedron, and the icosahedron are all regular polyhedra, but the shape made by gluing two tetrahedra together along a triangle is not, because some vertices have 4 triangles around them and others have 3. Prove that the five regular polyhedra mentioned in the previous sentence are the only regular polyhedra possible. (Hint: start your argument by using #7 and #9 above.) Regular polyhedra are also called Platonic solids.