

**National Bulgarian Mathematical Olympiad 1997**  
**Regional Round**

- (1) Find all positive integers  $a$ ,  $b$ , and  $c$  such that the roots of the equations

$$x^2 - 2ax + b = 0$$

$$x^2 - 2bx + c = 0$$

$$x^2 - 2cx + a = 0$$

are positive integers.

- (2) Let the convex quadrilateral  $ABCD$  be inscribed in a circle,  $F = AC \cap BD$  and  $E = AD \cap BC$ . If  $M$  and  $N$  are the midpoints of  $AB$  and  $CD$ , prove that

$$\frac{MN}{EF} = \frac{1}{2} \left| \frac{AB}{CD} - \frac{CD}{AB} \right|.$$

- (3) Prove that the equation

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0$$

has no solutions in rational numbers.

- (4) Find all continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x) = f(x^2 + \frac{1}{4})$  for all  $x \in \mathbb{R}$ .

- (5) Let  $K_1$  and  $K_2$  be unit squares with centers  $M$  and  $N$  such that  $MN = 4$ , two of the sides of  $K_1$  are parallel to  $MN$  and one of the diagonals of  $K_2$  lies on the line  $MN$ . Find the locus of midpoints of the segments  $XY$  where  $X$  and  $Y$  are interior points of  $k_1$  and  $k_2$ , respectively.

- (6) Find the number of all non-empty subsets of the set  $S_n = \{1, 2, \dots, n\}$  which do not contain two consecutive integers.

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## Final Round

(1) Consider the polynomials  $P_n(x) = \sum_{i=0}^k \binom{n}{3i+2} x^i$  where  $n \geq 2$  and  $k = \lfloor \frac{n-2}{3} \rfloor$ .

(a) Prove that  $P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$ .

(b) Find all integers  $a$  such that  $P_n(a^3)$  is divisible by  $3^{\lfloor \frac{n-2}{3} \rfloor}$  for all  $n \geq 2$ .

(2) Let  $M$  be the centroid of  $\triangle ABC$ . Prove the inequality

$$\sin \angle CAM + \sin \angle CBM \leq \frac{2}{\sqrt{3}}.$$

(a) if the circumcircle of  $\triangle AMC$  is tangent to the line  $AB$ ;

(b) for any  $\triangle ABC$ .

(3) Let  $n$  and  $m$  be positive integers and  $m+i = a_i b_i^2$  for  $i = 1, 2, \dots, n$ , where  $a_i$  and  $b_i$  are positive integers and  $a_i$  is square-free. Find all  $n$  for which there exists  $m$  such that  $a_1 + a_2 + \dots + a_n = 12$ .

(4) Let  $a$ ,  $b$ , and  $c$  be positive numbers such that  $abc = 1$ . Prove the inequality

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

(5) Let  $BM$  and  $CN$  be the angle bisectors in  $\triangle ABC$  and let ray  $MN \rightarrow$  intersect the circumcircle of  $\triangle ABC$  at point  $D$ . Prove that

$$\frac{1}{BD} = \frac{1}{AD} + \frac{1}{CD}.$$

(6) Let  $X$  be a set of  $n+1$  elements,  $n \geq 2$ . The ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  and  $(b_1, b_2, \dots, b_n)$  consisting of distinct elements of  $X$  are called "disjoint" if there exist distinct indices  $i$  and  $j$  such that  $a_i = b_j$ . Find the maximal number of  $n$ -tuples any two of which are "disjoint".