Even more BAMO preparation

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1. (BAMO 2006) All the chairs in a classroom are arranged in a square $n \times n$ array (in other words, n columns and n rows) and every chair is occupied by a student. The teacher decides to rearrange the students according to the following two rules:

   - Every student must move to a new chair.
   - A student can only move to an adjacent chair in the same row or to an adjacent chair in the same column. In other words, each student can move only one chair horizontally or vertically.

   (Note that the rules allow two students in adjacent chairs to exchange places.)

   Show that this procedure can be done if n is even, and cannot be done if n is odd.

2. (BAMO 2004) A given line passes through the center O of a circle. The line intersects the circle at points A and B. Point P lies in the exterior of the circle and does not lie on line AB. Using only an unmarked straightedge, construct a line through P, perpendicular to the line AB. Give complete instructions for the construction and prove that it works.

3. (BAMO 1999) Prove that among any 12 consecutive positive integers, there is at least one which is smaller than the sum of its positive divisors.

4. (BAMO 2002) Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to ABC. If M is the center of this square, find the measure of $\angle MBC$.

5. (BAMO 2005) Prove that if two medians of a triangle are the same length, then the triangle is isosceles.

6. (BAMO 2000) Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1. (Two integers are relatively prime if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus $37 = 22 + 15$ represents the number 37 in the desired way.)

7. (BAMO 2003) An integer $n > 1$ has the following property: for every (positive) divisor d of n, $d + 1$ is a divisor of $n + 1$. Prove that n is prime.

8. (BAMO 2007) Two sequences of positive integers, $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are given, such that $y_{n+1}/x_{n+1} > y_n/x_n$ for each $n \geqslant 1$. Prove that there are infinitely many values of n such that $y_n > \sqrt{n}$.

9. (BAMO 2001) For each positive integer n, let a_n be the number of permutations τ of $\{1, 2, \ldots, n\}$ such that $\tau(\tau(\tau(x))) = x$ for $x = 1, 2, \ldots, n$. The first few values are

$$a_1 = 1, a_2 = 1, a_3 = 3, a_4 = 9.$$

Prove that 3^{334} divides a_{2001}.