CLASSICAL THEOREMS IN PLANE GEOMETRY

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Note: All objects in this handout are *planar* - i.e. they lie in the usual plane. We say that several points are *collinear* if they lie on a line. Similarly, several points are *concyclic* if they lie on a circle; an *inscribed* (cyclic) polygon has its vertices lying on a circle. If three distinct points A, B and C are collinear, then the *directed ratio* $\overline{AB}/\overline{CB}$ is the ratio of the lengths of segments AB and CB, taken with a sign "+" if the segments have the same direction (i.e. B is *not* between A and C), and with a sign "−" if the segments have opposite directions (i.e. B is between A and C). Several objects (lines, circles, etc.) are *concurrent* if they all intersect in some point.

1. (Menelaus) Let A_1, B_1 and C_1 be three points on the sides BC, CA and AB of $\triangle ABC$. Prove that they are collinear (cf. Fig. 1) iff
$$\frac{\overline{AB_1}}{\overline{CB_1}} \cdot \frac{\overline{CA_1}}{\overline{BA_1}} \cdot \frac{\overline{BC_1}}{\overline{AC_1}} = 1.$$

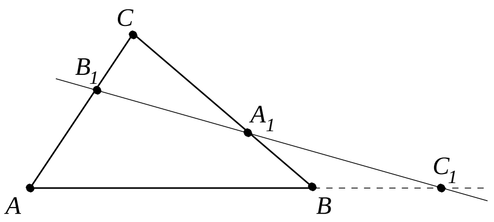

FIGURE 1

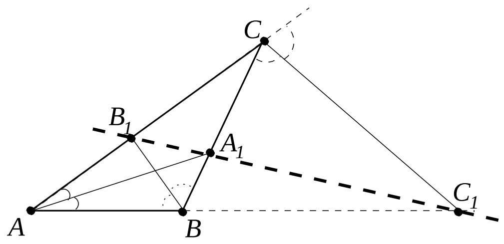

FIGURE 2A

2. (a) Prove that the interior angle bisectors of two angles of nonisosceles $\triangle ABC$ and the exterior angle bisector of the third angle intersect the opposite sides (or their continuations) of $\triangle ABC$ in three collinear points. (cf. Fig. 2a)
   (b) Prove that the exterior angle bisectors of nonisosceles $\triangle ABC$ intersect the continuations of opposite sides of $\triangle ABC$ in three collinear points.
   (c) Prove that the tangents at the vertices of nonequilateral $\triangle ABC$ to the circumcircle of $\triangle ABC$ intersect the continuations of opposite sides of $\triangle ABC$ in three collinear points. (cf. Fig. 2c)
3. (Pascal) If the hexagon $ABCDEF$ is cyclic and its opposite sides, AB and DE, BC and EF, CD and FA, are pairwise not parallel, prove that their three points of intersection, X, Y and Z, are collinear. (cf. Fig. 3)

   The same statement is true if the circle is replaced by an ellipse, hyperbola or parabola.[1] The statement is also true if some of the vertices of the hexagon coincide

[1] These all are *conics*, i.e. *projectively equivalent* to a circle.

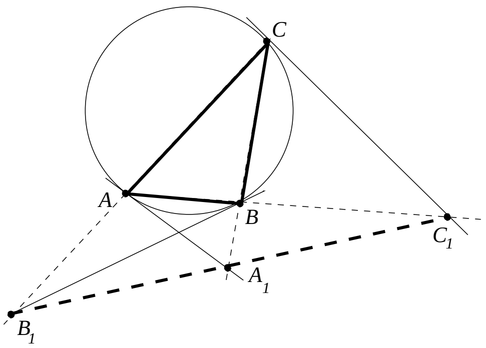

FIGURE 2C

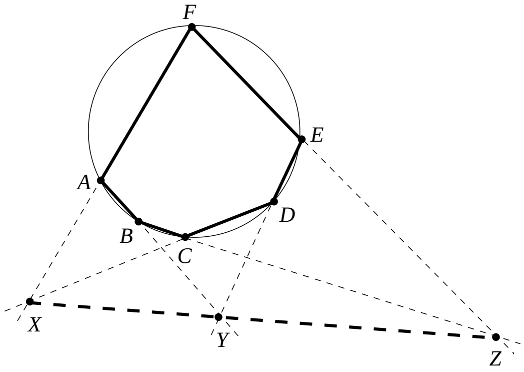

FIGURE 3

– then replace the corresponding side of the hexagon by the tangent to the circle at the corresponding vertex. Thus, obtain the following:

(a) If $A = B$, $C = D$, $D = F$, deduce to Problem 2c.
(b) If $E = F$, formulate the property of any inscribed pentagon.
(c) If $A = F$ and $D = E$, for the inscribed quadrilateral $ABCD$ we have: the intersection points of AB and the tangent at D, of CD and the tangent at A, and of BC and AD, are collinear.
(d) If $A = F$ and $C = D$, the intersection points of the pairs of opposite sides of an inscribed quadrilateral and the intersection of the tangents at two opposite vertices are collinear. (Actually, the tangents at *any* pair of opposite vertices should also work.)

4. (Desargues) $\triangle ABC$ and $\triangle A_1B_1C_1$ are positioned in such a way that lines AA_1, BB_1, and CC_1 intersect in a point O. If lines AB and A_1B_1, AC and A_1C_1, BC and B_1C_1 are pairwise not parallel, prove that their points of intersection, L, M and N, are collinear. (cf. Fig. 4)

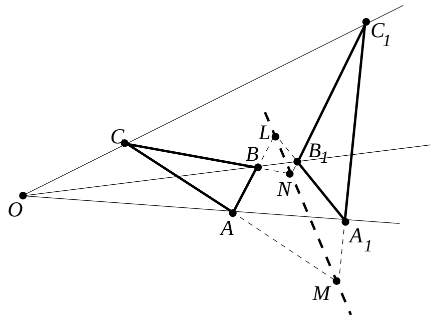

FIGURE 4

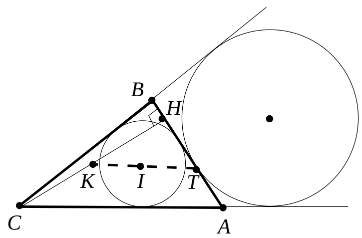

FIGURE 5

Note: The *incircle* of $\triangle ABC$ is the circle inscribed in $\triangle ABC$ (i.e. tangent to all three sides of the triangle.) Its center is called the *incenter* of $\triangle ABC$; it lies on the angle bisectors of $\triangle ABC$. The *excircle* of $\triangle ABC$ tangent to side AB is the circle tangent to side AB and to the *extentions* of sides BC and AC. Its center is called an *excenter* of $\triangle ABC$. On what bisectors does this excenter lie? The *circumcircle* of $\triangle ABC$ is the circle passing through the vertices A, B and C. Its center is called the *circumcenter* of $\triangle ABC$; it lies on the perpendicular bisectors of the sides of the triangle.

5. Prove that the midpoint K of the altitude CH in $\triangle ABC$, the incenter I of $\triangle ABC$, and the tangency point T on AB of the excircle of $\triangle ABC$ (tangent to side AB) are collinear. (cf. Fig. 5)

6. (Gauss's line with respect to l) Line l intersects the sides (or continuations of) BC, CA and AB of $\triangle ABC$ in points P_1, P_2 and P_3. Prove that the midpoints M_1, M_2 and M_3 of AP_1, BP_2 and CP_3 are collinear. (cf. Fig. 6)

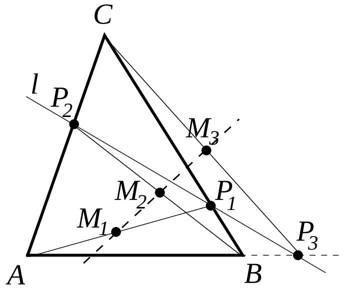

FIGURE 6

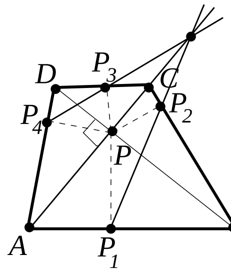

FIGURE 7

7. Let $ABCD$ be a quadrilateral with perpendicular diagonals intersecting in P. The feet of the perpendiculars from P to sides AB, BC, CD and DA are P_1, P_2, P_3 and P_4. Prove that lines P_1P_2, P_3P_4 and CA are concurrent. (cf. Fig. 7)

8. (Simpson) Prove that the feet of the perpendiculars dropped from a point M on the circumcircle k of $\triangle ABC$ to the sides of the triangle are collinear. More generally, let S be the area of $\triangle ABC$, R – the circumradius, and d – the radius of a circle ϵ concentric to k. Let A_1, B_1 and C_1 be the feet of the perpendiculars dropped from an arbitrary point on ϵ to the sides of $\triangle ABC$. Prove that the area S_1 of $\triangle A_1B_1C_1$ is given by the formula $S_1 = \frac{1}{4}S\left|1 - \frac{d^2}{R^2}\right|$. In particular, when $\epsilon = k$, then $S_1 = 0$, and hence A_1, B_1 and C_1 are collinear. (cf. Fig. 8)

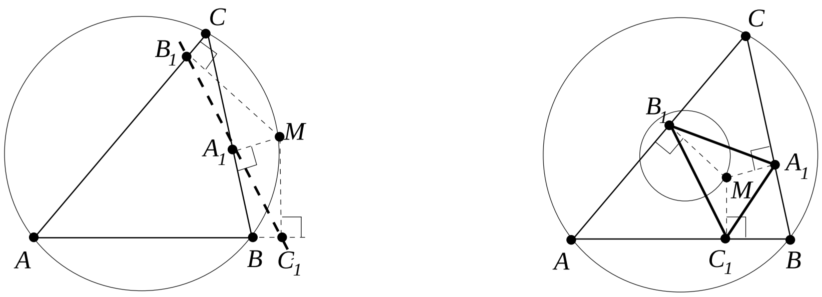

FIGURE 8

9. (Salmon) Through a point M on a circle ϵ draw three arbitrary chords MA, MB and MC, and using each chord as a diameter, draw three new circles ϵ_1, ϵ_2, and ϵ_3. Prove that the pairwise intersections of the ϵ_i's (other than M) are collinear.

Note: Let H be the orthocenter of $\triangle ABC$ (i.e. the intersection of the altitudes of the triangle.) The *Euler circle of 9 points* for $\triangle ABC$ is the circle passing through the midpoints of the sides of $\triangle ABC$, the midpoints of AH, BH and CH, and the feet of the altitudes of $\triangle ABC$. In fact, the center of this circle is the midpoint of HO (O is the circumcenter of $\triangle ABC$), and its radius is half of the circumradius of $\triangle ABC$. Why? (cf. Fig. 10a)

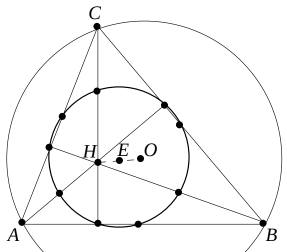

FIGURE 10A

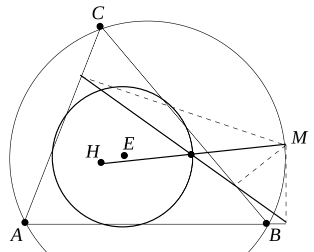

FIGURE 10B

10. Prove that Simpson's line of $\triangle ABC$ with respect to point M on the circumscribed circle k of $\triangle ABC$, line MH where H is the orthocenter of $\triangle ABC$, and the Euler circle of 9 points for $\triangle ABC$ are concurrent. (cf. Fig. 10b)
11. (Ceva) Let A', B' and C' be three points on the sides (or continuations of) BC, CA, AB of $\triangle ABC$. Prove that AA', BB', CC' are concurrent or are parallel iff
$$\frac{\overline{AB'}}{\overline{CB'}} \cdot \frac{\overline{CA'}}{\overline{BA'}} \cdot \frac{\overline{BC'}}{\overline{AC'}} = -1.$$
12. (Gergonne's point) Prove that the lines connecting the vertices of a triangle with the points of tangency of the inscribed circle are concurrent. (cf. Fig. 12)

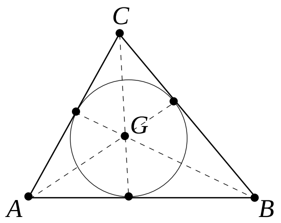

FIGURE 12

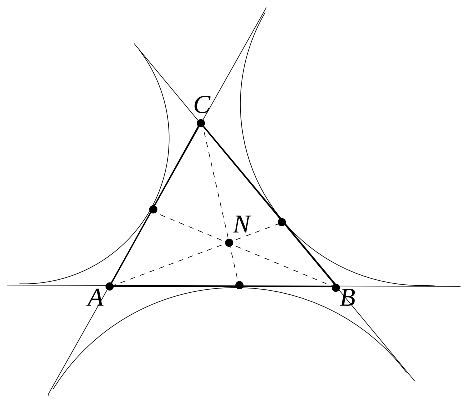

FIGURE 13

13. (Nagel's point) Prove that the lines connecting the vertices of a triangle with the corresponding points of tangency of the three externally inscribed circles are concurrent (cf. Fig. 13.) Note that these are also the three lines through the vertices of the triangle and dividing each its perimeter into two equal parts.
14. Let M be an arbitrary point on side AB of $\triangle ABC$. Let P and Q be the intersection points of the angle bisectors of $\angle BMC$ and $\angle AMC$ with sides BC and AC, respectively. Prove that lines AP, BQ and CM are concurrent.

15. Let A_1, B_1, C_1 be points on the sides of an acuteangled $\triangle ABC$ so that the lines AA_1, BB_1 and CC_1 are concurrent. Prove that CC_1 is an altitude in $\triangle ABC$ iff it is the angle bisector of $\angle B_1C_1A_1$.

16. In the acuteangled $\triangle ABC$ a semicircle k with center O on side AB is inscribed. Let M and N be the points of tangency of k with sides BC and AC. Prove that lines AM, BN and the altitude CD of $\triangle ABC$ are concurrent. (cf. Fig. 16)

17. A circle k intersects side AB of $\triangle ABC$ in C_1 and C_2, side CA – in B_1 and B_2, side BC – in A_1 and A_2. The order of these points on k is: A_1, A_2, B_1, B_2, C_2, C_1. Prove that lines AA_1, BB_1, CC_1 are concurrent iff AA_2, BB_2, CC_2 are concurrent. (cf. Fig. 17)

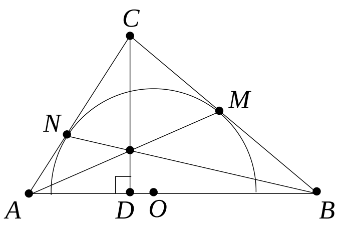

FIGURE 16

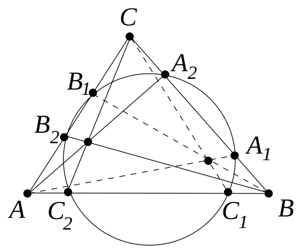

FIGURE 17

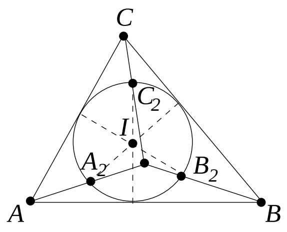

FIGURE 18

18. Let the points of tangency of the incircle of $\triangle ABC$ with the sides AB, BC and CA be C_1, A_1 and B_1, and let A_2, B_2 and C_2 be their reflections across the incenter I of $\triangle ABC$. Prove that lines AA_2, BB_2 and CC_2 are concurrent. (cf. Fig. 18)

19. (Gauss) If the two pairs of opposite sides of a quadrilateral $ABCD$ intersect in E and F, prove that the midpoint N of EF lies on the line through the midpoints L and M of the diagonals AC and BD. (cf. Fig. 19)

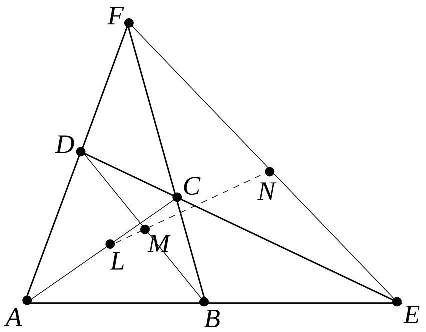

FIGURE 19

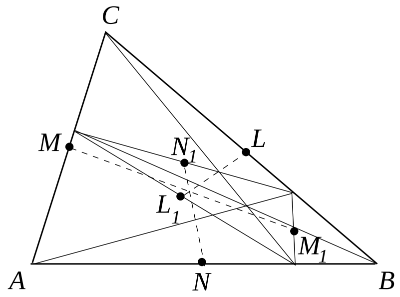

FIGURE 20

20. Point P lies inside $\triangle ABC$. Lines AP, BP, CP intersect the sides BC, CA, AB in A_1, B_1, C_1, respectively, and L, M, N, L_1, M_1, N_1 are the midpoints of the segments $BC, CA, AB, B_1C_1, C_1A_1, A_1B_1$. Prove that LL_1, MM_1 and NN_1 are concurrent. (cf. Fig. 20)

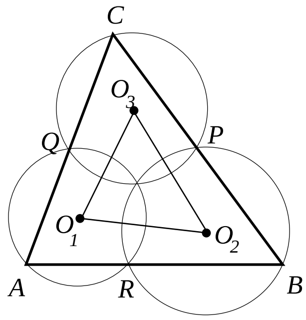

FIGURE 21

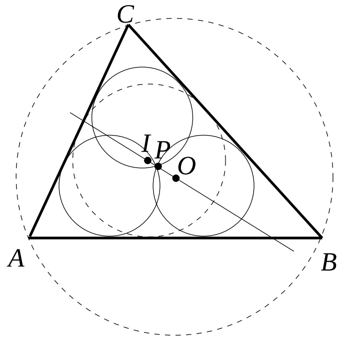

FIGURE 22

21. Let P, Q and R be points on the sides BC, CA and AB of $\triangle ABC$. Let O_1, O_2 and O_3 be the circumcenters of $\triangle AQR$, $\triangle BRP$ and $\triangle CPQ$. Prove that $\triangle O_1O_2O_3 \sim \triangle ABC$. (cf. Fig. 21)
22. (IMO'81) Three congruent circles pass through point P inside $\triangle ABC$. Each circle is inside $\triangle ABC$ and is tangent to two of its sides. Prove that the circumcenter O and incenter I of $\triangle ABC$ and P are collinear. (cf. Fig. 22)
23. (Brianchon) If the hexagon $ABCDEF$ is circumscribed around a circle, prove that its three diagonals AD, BE and CF are concurrent. (cf. Fig. 23)

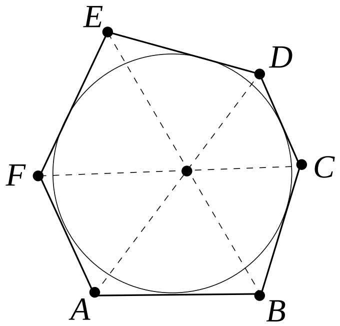

FIGURE 23

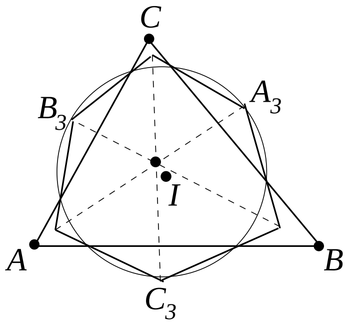

FIGURE 24

24. (Saint Petersburg Olympiad) Point I is the incenter of $\triangle ABC$. Some circle with center I intersects side BC in A_1 and A_2, side CA in B_1 and B_2, and side AB in C_1 and C_2. The six points obtained in this way lie on the circle in the following order: A_1, A_2, B_1, B_2, C_1, C_2. Points A_3, B_3 and C_3 are the midpoints of the arc A_1A_2, B_1B_2 and C_1C_2 respectively. Lines A_2A_3 and B_1B_3 intersect in C_4, lines B_2B_3 and C_1C_3 – in A_4, and lines C_2C_3 and A_1A_3 – in B_4. Prove that the segments A_3A_4, B_3B_4 and C_3C_4 intersect in one point. (cf. Fig. 24)
25. (Bulgarian IMO Test '08) In $\triangle ABC$ let AM ($M \in BC$) be a median and let CC_1 ($C_1 \in AB$) and BB_1 ($B_1 \in AC$) be two altitudes. The line through A perpendicular to AM intersects lines BB_1 and CC_1 in points E and F, respectively. Denote by k the circumcircle of $\triangle EFM$. Let k_1 and k_2 be circles tangent to both EF and to the arc EF on K not containing M. If P and Q are the intersection points of k_1 and k_2, prove that points P, Q, and M lie on a line.

26. (IMO '08, Spain) An acute-angled $\triangle ABC$ has orthocenter H. The circle passing through H with center the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with center the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with center the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, and C_2 lie on a circle.

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