Symmetry

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Warm-up problems

1. Players take turns putting pennies on a round table. The pennies cannot overlap; they can overhang the table but should not fall off. The player unable to place a penny loses. Which of the two players has a winning strategy?
2. A ranger standing at a point *A* above a straight river notices a fire blazing at a point *B* on the same side of the river. Find the shortest path for the ranger to get a bucket of water from the river and then come to the point *B* to extinguish the fire.
3. A cue ball is at the side of a rectangular pool table. Find a direction in which the ball must be hit in order to bounce off three sides of the table and return to its original position.
4. Find all values of *a* and *b* such that the given system of equations has exactly one solution.

$$\begin{cases} x+y+z=3 \\ x^2+y^2+z^2=a \\ x^3+y^3+z^3=b \end{cases}$$

Symmetric Functions and Elementary Symmetric Polynomials

A symmetric function is a function such that it remains unchanged under any permutations of the variables. For example, $P(x,y)=x^5-3x^2y^2+y^5$, or

$$Q(x_1,x_2,x_3)=\frac{x_1+x_2}{x_1x_2x_3}+\frac{x_2+x_3}{x_1x_2x_3}+\frac{x_3+x_1}{x_1x_2x_3}.$$

Among symmetric functions there's an important class called elementary symmetric polynomials. They are defined as follows.

$\sigma_1(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$

$\sigma_2(x_{1,} x_2, \ldots, x_n) = x_1 x_2 + x_1 x_2 + \ldots + x_n x_{n-1}$ (the sum of the products of all possible pairs of variables)

$\sigma_3(x_1, x_2, \ldots x_n) = x_1 x_2 x_3 + x_1 x_2 x_4 + \ldots + x_n x_{n-1} x_{n-2}$ (the sum of the products of all possible triples)

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$\sigma_n = x_1 x_2 \ldots x_n$

These functions are important because they are building blocks in the realm of all symmetric functions. They are also related to the roots of polynomials. The following two theorems make these statements a bit clearer.

Theorem 1: Every symmetric polynomial function (rational function) is a polynomial function (rational function) of elementary symmetric polynomials.

Theorem 2: Let $X_1, X_2, \ldots X_n$ be the roots of the polynomial equation $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$.

Then $\sigma_k(X_1, X_2, \ldots, X_n) = (-1)^k a_k, \quad k = 1, 2, \ldots, n$.

Additional problems.

1. There are several minuses written along a line. A player replaces either one minus by a plus or two adjacent minuses by two pluses. The player who replaces the last minus wins. Which of the two players has a winning strategy?
2. Same game as above, only the minuses are written around a circle.
3. Two rivers meet, forming a Y shape. On the peninsula between them are two points, *A* and *B*. What is the shortest route a person can take from *A* to *B*, getting a bucket of water from each river in between?
4. In a plane, a line separates points *A* and *B*. Find a point *P* of the line such that the difference between the segments *AP* and *BP* is as large as possible.
5. Find all solutions of the given system of polynomial equations.

$$\begin{cases} x^2 + y^2 = 3 \\ x^3 + y^3 = 4 \end{cases}$$

6. Given that $x + y + z = 1, x^2 + y^2 + z^2 = 2, x^3 + y^3 + z^3 = 3$, find $x^4 + y^4 + z^4$.
7. Solve the given system of polynomial equations.

$$\begin{cases} x + y + z = 0 \\ x^2 + y^2 + z^2 = 6ab \\ x^3 + y^3 + x^3 = 3(a^3 + b^3) \end{cases}$$

Bonus problem

For an acute triangle ABC, find an inscribed triangle with a minimal perimeter.