Generating Functions, Infinite Series, and the Relation between the Discrete and Continuous (with little or no Calculus...)

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During this talk we will explore some properties of infinite series and generating functions. The exercises below are representative of problems solvable using techniques we will discuss. Solutions and hints will be provided during the talk.

Geometric Series

1. Find the rational number whose decimal expansion is .12357357357 …

2. Find the exact time, in hours/minutes/seconds, at which the hands of a clock first make a straight angle after 12:00.

3. Find the exact time, in hours/minutes/seconds, at which the hands of a clock first make a right angle after 3:00.

Fibonacci Numbers

1. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^n}{f_n f_{n+1}} = \frac{1}{\tau^2}$

2. Prove that $\sum_{n=1}^{\infty} \frac{f_n}{f_{n+1} f_{n+2}} = 1$

3. Prove that $\sum_{k=0}^{n} \binom{n}{k} f_k = f_{2n}$

Infinite Series

4. Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

5. Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

6. Prove that $\sum_{n=0}^{\infty} \frac{(-1)^n (2n+3)}{(n+1)(n+2)} = 1$.

7. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$.

8. Prove that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots = \frac{\pi^2}{8}$

9. Use Euler's trick to prove that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Recursion Relations

10. Solve the relation $a_{n+1} = 2\,a_n + 2$ for $n = 1, 2, 3, 4, \ldots,$ given that $a_1 = 2$.

11. Solve the relation $a_{n+1} = 2\,a_n + n$ for $n = 1, 2, 3, 4, \ldots,$ given that $a_0 = 1$.

12. Use generating functions to solve the relation $t_n = t_{n-1} + 2\,t_{n-2}$, with $t_1 = 1$ and $t_2 = 0$.

13. Use generating functions to find an explicit formula for the Fibonacci numbers $1, 1, 2, 3, 5, 8, \ldots$

Combinatorics

13. Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2\,n}{n}$

14. Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^k$

15. Prove that $\displaystyle\sum_{k=1}^{n} k \binom{n}{k} = n\,2^{n-1}$

16. Prove that $\displaystyle\sum_{k=1}^{n} k(k-1) \binom{n}{k} = n(n-1)\,2^{n-2}$

Partitions

17. The number of partitions of n in which no part occurs more often than d times is the same as the number of partitions of n in which no term is a multiple of $(d + 1)$.

18. The number of partitions of n in which no part appears exactly once is the same as the number of partitions of n in which no part is congruent to 1 or 5 modulo 6.

19. The number of partitions of n in which no even part is repeated is the same as the number of partitions of n in which no part occurs more often than 3 times, and is also the number of partitions of n in which no part is divisible by 4.

20. Evaluate the product $(1 + x)\left(1 + x^2\right)\left(1 + x^4\right)\left(1 + x^8\right)\left(1 + x^{16}\right)\left(1 + x^{32}\right)\left(1 + x^{64}\right)$

21. Evaluate the product $\left(1 + x + x^2\right)\left(1 + x^3 + x^6\right)\left(1 + x^9 + x^{18}\right)\left(1 + x^{27} + x^{54}\right)\left(1 + x^{81} + x^{162}\right)$

Miscellaneous

22. Find any possible "non-standard" dice, i.e. whose faces are not 1-6, but whose expected values agree with standard dice.

23. What is the decimal expansion of $\dfrac{1}{(1-x)^2}$ at $x = .1$? At $x = .01$?

24. What is the decimal expansion of $\dfrac{1}{1 - x - x^2}$ at $x = .1$? At $x = .01$?

25. What is the sum $\frac{1}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \frac{5^2}{2^5} + \cdots + \frac{n^2}{2^n} + \cdots$?

Sources

Mathematical Gems (I, II and III), by Ross Honsberger (MAA, Dolciani Mathematical Expositions)

Excursions in Calculus, by Robert Young (MAA, Dolciani Mathematical Expositions, Number 13)

The Art and Craft of Problem Solving, by Paul Zeitz (Wiley publishers)