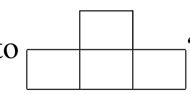
Berkeley Math Circle
BAMO 2009 Preparation – Ivan Matić

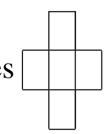
1. The rectangle $MNPQ$ is inside the rectangle $ABCD$. The portion of the rectangle $ABCD$ outside of $MNPQ$ is colored in green. Using just a straightedge construct a line that divides the green figure in two parts of equal areas.

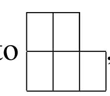
2. Given a triangle ABC such that $\angle B = 90^\circ$, denote by k a circle with center on BC that is tangent to AC. Denote by T a point of tangency of k and the tangent from A to k (different from AC). If B' is the midpoint of AC and M the intersection of BB' and AT, prove that $MB = MT$.

3. If a, b, c are positive real numbers prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

4. If a, b, c are positive real numbers prove that
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

5. Is it possible to tile 8×8 board with figures congruent to ? What about 10×10 board?

6. Determine the maximal number of figures  that can be placed in a grid 6×6 without overlaping.

7. If a $5 \times n$ rectangle can be tiled with n pieces congruent to , prove that n is even.

8. Prove that for each $n \geq 3$, there are positive odd integers x and y such that
$$2^n = x^2 + 7y^2.$$

9. There are 12 students in a class and among them each two are either friends or enemies. Prove that it is possible to choose two and paint them in green such that the following condition is satisfied: Among the remaining 10 students, it is possible to choose 5 each of which is either mutual friend or mutual enemy to both of green students.

10. If a, b, c are positive real numbers prove that
$$\frac{a+b}{2b+c} + \frac{b+c}{2c+a} + \frac{c+a}{2a+b} \geq 2.$$

11. The incircle of triangle ABC touches sides BC, CA, and AB at A_1, B_1, and C_1 respectively. Prove that the perpendiculars from the midpoints of A_1B_1, B_1C_1, and C_1A_1 to AB, BC, and CA are concurrent.

12. A circle with center O passes through points A and C and intersects the sides AB and BC of the triangle ABC at points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at two distinct points B and M. Prove that $\angle OMB = 90^\circ$.