Mathematical Induction

Berkeley Math Circle

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Principle of Mathematical Induction: Let S_n be a statement involving the positive integer n. Then S_n will be true for every positive integer provided the following two conditions:

1. S_1 is true.

2. S_n being true implies S_{n+1} is true.

Problems

1) Prove that $n < 2^n$.

2) Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

3) Prove that 5 divides $6^n - 1$.

4) Find a formula for $1 + 5 + 9 + \cdots + (4n - 3)$ and prove it.

5) Prove $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.

6) Find a formula for $2^0 + 2^1 + 2^2 + \cdots + 2^n$ and prove it.

7) Find a formula for $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{n}\right)$ and prove it.

8) Prove $\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$. Use this and problem **(2)** to find a formula for $1^2 + 2^2 + \cdots + n^2$.

9) Prove $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$ for $0 \le k \le n$.

10) Prove $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

11) Prove that the Fibonacci number $F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$, where ϕ is the golden ratio, $\frac{1+\sqrt{5}}{2}$.

12) Prove $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$.

13) Find n if $\frac{1^3+3^3+5^3+\cdots+(2n-1)^3}{2^3+4^3+6^3+\cdots+(2n)^3} = \frac{199}{242}$.

14) Show $n^n > 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)$.

15) Find a formula for $2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + \cdots + (3n-1)(3n+2)$ and prove it.

16) Find a formula for $1 \cdot 2^2 \cdot 3 + 2 \cdot 3^2 \cdot 4 + \cdots + n(n+1)^2(n+2)$ and prove it.

17) Find a formula for $\frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)(2n+3)}$ and prove it.

18) Find a formula for $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)}$ and prove it.

19) The linear recursive sequence A_n is defined by $A_1 = 1$, $A_2 = 5$, and $A_n = 21A_{n-2} - 4A_{n-1}$ for $n \geq 3$. Find a formula for A_n and prove it.

20) A set of n dominos, each domino 1 inch by 2 inches by $\frac{1}{4}$ inch, is piled into a tower, each domino laying flat. What is the furthest the top domino can overhang the bottom one without the tower collapsing?