BAMO ADVANCED PREPARATION SECTION

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1. First Hour

1.1. Simpler Problems.

Problem 1.1.1. Prove that $m(m+1)$ cannot be a power (greater than 1) of an integer for any natural m.

Problem 1.1.2. Let f be a polynomial with integer coefficients and roots $\alpha_1, \ldots, \alpha_n$. Let $M = \max_i |\alpha_i|$. Prove that if for some x_0, such that $|x_0| > M + 1$, $f(x_0)$ is a prime number, then f is a prime polynomial (cannot be factored into polynomials with integer coefficients and of smaller degree)

Problem 1.1.3. Can all vertices of an equilateral triangle have integer coordinates?

Problem 1.1.4. Proof that in any triangle ABC, angle bisector AE lies between meridian AM and altitude AH.

1.2. Harder Problems.

Problem 1.2.1. Cauchy Theorem: Let $f(x) = x^n - b_1x^{n-1} - \ldots - b_n$, where b_i are non-negative and at least one of them is positive. Prove that f has a single (counting the multiplicity) positive root. Moreover, the absolute values of all other roots are less than it.

Problem 1.2.2. Let $A_1A_2 \ldots A_{2m}$ be a convex $2m$-gon. Point P is taken inside it such that it does not lie on any diagonal. Prove that P lies in an even number of triangles with vertices in points $A_1, A_2, \ldots, A_{2m}$.

Problem 1.2.3. Is there a closed broken line with odd number of segments of equal length each vertex of which has integer coordinates?

2. Second Hour

2.1. Simpler Problems.

Problem 2.1.1. Natural numbers a and b are coprime. Prove that the greatest common divisor of $a+b$ and $a^2 + b^2$ is either 1 or 2.

Problem 2.1.2. There are 3 non-colinear pucks on the ground. A hockey player always hits a puck such that it goes between the other two and does not stop of the line formed by them. Can the player after 25 hits return each puck to its original position.

Problem 2.1.3. Given a rectangle $ABCD$ and 4 circles C_1, C_2, C_3, C_4 with centers at A, B, C, D and radii r_1, r_2, r_3, r_4. Moreover, $r_1 + r_3 = r_2 + r_4 < d$, where d is the diagonal of the rectangle. Two pairs of external tangents are drawn to circles C_1, C_3 and C_2, C_4. Prove that one can inscribe a circle into the quadrilateral formed by these 4 lines.

Problem 2.1.4. Are there natural numbers x and y such that $x^2 + y$ and $x + y^2$ are squares of natural numbers.

2.2. **Harder Problems.**

Problem 2.2.1. In triangle ABC sides AC and BC are not equal. Prove that angle bisector CE of angle $\angle ACB$ equally bisects the angle between the meridian CM and altitude CH if and only if, angle $\angle ACB$ is 90^o

Problem 2.2.2. Prove that for any three infinite sequences of natural numbers $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$, there exist two indices p and q such that $a_p \geq a_q$, $b_p \geq b_q$, and $c_p \geq c_q$.

Problem 2.2.3. Nine out of 100 cells of a 10x10 board are occupied by a Marcian bacteria. Each day it spreads to all cells that have at least two neighboring cell (that sharing an edge) already occupied. Prove that it will never occupy all the cells. What if initially 10 cells are occupied?

Problem 2.2.4. Suicidal King: On a $1000x1000$ chess board, there is a black king and 499 white rocks. Prove that the king can always move in such a way as to be killed by a rock, independently of the initial configuration.

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