All about (regular) n-gons

Andrew Dudzik

December 16, 2008

The Basics

What are the interior angles of a regular n-gon?

How many diagonals does a regular n-gon have?

Trigonometry

What is the area of a regular n-gon inscribed in the unit circle? What happens to this number as n gets very large?

Three distinct points are chosen at random from the vertices of a regular n-gon inscribed in the unit circle. What is the expected area of the triangle they form?

Combinatorics

How many intersection points are formed when we draw all the diagonals of a regular n-gon, if we can bend the diagonals a little so that no more than two intersect at a point?

How many regions are formed when we draw all the diagonals of a regular n-gon, bent as before?

Complex Numbers

What is the product of the lengths of all sides and diagonals emanating from a single vertex of a regular n-gon inscribed in the unit circle?

Symmetries

How many symmetries does a regular n-gon have in two dimensions? Three dimensions? Four dimensions?

1. A ravenous goat is tethered to one corner of a grassy n-gon-shaped pen. The length of the tether and the length of the sides of the pen are each r. Find the total area of grass over which the goat can graze.

2. Given five curmudgeonly goats in a square pen of side length 1, what is the maximum length d such that the goats can arrange themselves with no goat less than distance d from any other?

3. How about seven cantankerous goats in a regular hexagon of side length 1?

4. n irascible goats are standing at the vertices of a regular n-gon with side length 1 furlong. All of them suddenly become enraged and each begins to charge directly at the goat immediately counterclockwise, at a speed of one furlong per fortnight. Will the goats eventually collide, and if so, how long will it take?

5. What are all the values of n for which you can tile the plane with regular n-gons of varying size?

6. The sum of the lengths of all sides and diagonals emanating from a vertex of a regular n-gon inscribed in the unit circle is $2cot(\pi/(2n))$. Use this fact to find the average length of a chord of the unit circle.

7. Color n vertices of a $2n$-gon red, and the others blue. Let R be the set of all distances between two red vertices, and B the set of all distances between two blue vertices. Show that $R = B$.

8. The UK 50 pence coin is a 7-sided figure. Its shape can be constructed from the vertices of a regular heptagon as follows: connect each pair of adjacent vertices by an arc of the circle with center at the opposite vertex. This coin has the strange property that its diameter is the same when measured from any direction! If its diameter is d, show that the circumference of the coin is πd.

9. n people are at a restaurant, sitting at the vertices of an n-gon-shaped dinner table. Their orders have been mixed up—in fact, none of them have received the correct entree. Show that the table may be rotated so that at least two people are sitting in front of the correct entree.

10. (Romania 1995) Find the number of ways of coloring the vertices of a regular n-gon with p colors, such that no two adjacent vertices have the same color.

11. Does there exist a regular n-gon such that exactly half of its diagonals are parallel to one of its sides?

12. (Putnam 2005) Let $n \geqslant 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), \ldots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n - 1$.

13. Show that the sum of the squares of the lengths of all sides and diagonals emanating from a vertex of a regular n-gon inscribed in the unit circle is $2n$.

14. (Russia 1993) Given a regular $2n$-gon, show that we can assign to each side and diagonal a vector pointing from one to the other, such that the sum of all such vectors is zero.

15. (Russia 1995) All sides and diagonals of a regular 12-gon are painted in 12 colors. (each segment is painted in one color) Is it possible that for any three colors there exist three vertices which are joined with each other by segments of these colors?

16. (Iran 2008) Let P be a regular polygon. A regular sub-polygon of P is a subset of vertices of P with at least two vertices that divides the circumcircle into equal arcs. Show that there is a subset of vertices of P whose intersection with each sub-polygon of P contains an even number of vertices.

17. (USAMO 1997) To *clip* a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them with three segments AM, MN, NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon P_6 of area 1 is clipped to obtain a heptagon P_7. Then P_7 is clipped (in one of the seven possible ways) to obtain an octagon P_8 and so on. Prove that no matter how the clippings are done, the area of P_n is greater than 1/3 for all $n \geqslant 6$.

18. (USAMO 2008) Let P be a convex polygon with n sides, $n \geqslant 3$. Any set of $n-3$ diagonals that do not intersect in the interior of a polygon determine a *triangulation* of P into $n-2$ triangles. If P is regular and there is a triangulation of P consisting only of isoceles triangles, find all possible values of n.

19. (Miklos Schweitzer, 1967) Let $\sigma(S_n, k)$ denote the sum of k-th powers of the lengths of the sides of the convex n-gon S_n inscribed in the unit circle. Show that for any natural number greater than 2 there exists a real number k_0 between 1 and 2 such that $\sigma(S_n, k_0)$ achieves its maximum for the regular n-gon.

20. (Miklos Schweitzer, 1973) Let $v_1, \dots v_7$ be the vertices of a regular heptagon inscribed in the unit circle, and $S_1, \dots S_7$ be connected regions in the plane with diameter less than 1 such that $v_1, v_2 \in S_1$, $v_2, v_3 \in S_2 \dots$ and $v_7, v_1 \in S_7$. If T is a connected region of diameter greater than 4 such that $0 \in T$, show that T intersects some S_i.

21. How many intersection points are formed when we draw all the diagonals of a regular n-gon? (no double-counting this time!)