Berkeley Math Circle
Monthly Contest 2
Due November 4, 2008

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 2
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. A building has seven rooms numbered 1 through 7, all on one floor, and any number of doors connecting these rooms. These doors may be one-way, admitting motion in only one of the two directions, or two-way. In addition, there is a two-way door between room 1 and the outside, and a treasure in room 7. Your object is to choose the arrangement of the rooms and the locations of the doors in such a way that

   (a) it is possible to enter room 1, reach the treasure, and make it back outside,

   (b) the minimum number of steps required to to this (each step consisting of walking through a door) is as large as possible.

2. Prove that there is exactly one way to place circles in four of the blank squares of the cross-equation puzzle at right such that, no matter what natural numbers are placed in the circled squares, the five uncircled squares can be filled with natural numbers that make the three horizontal and three vertical equations true.

	+		=	
+	■	=	■	+
	=		+	
=	■	+	■	=
	+		=	

3. A number is called a *j-half* if it leaves a remainder of j when divided by $2j+1$.

   (a) Prove that for any k, there is a number which is simultaneously a j-half for $j = 1, 2, \ldots, k$.

   (b) Prove that there is no number which is a j-half for all positive integers j.

4. Let AOB be a 60-degree angle. For any point P in the interior of $\angle AOB$, let A' and B' be the feet of the perpendiculars from P to AO and BO respectively. Denote by r and s the distances OP and $A'B'$. Find all possible pairs of real numbers (r, s).

5. Prove that for every positive integer n, there is an integer x such that $x^2 - 17$ is divisible by 2^n.